Lecture 12

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ECE 336
Stochastic Models for the Analysis of Computer Systems and Communication Networks

Outline of Lecture 12

- Stabilization of Slotted Aloha
- Splitting Algorithms
- Introduction to CSMA

We know that as the number of users (nodes), $m \to \infty$, slotted Aloha becomes unstable.

Ordinary slotted aloha is unstable for any arrival rate ρ >0 (recall ρ is normalized to sloth length)

Maximization stable throughput of slotted Aloha = 0
i.e., the least upper bound of arrival rates for which the system is stable

The questions we would like to address are the following:

1) Suppose have m users, $m \to \infty$ and *n* bocklogged packets, then

$$G(n) = (m-n)\sigma + n\sigma = \rho + n\gamma \text{ as } m \to \infty$$

$$\sigma = \text{prob of new arrival in a slot}$$

$$\gamma = \text{retransm prob}$$

$$know \;\; \rho \approx Ge^{-G} \quad \text{and } \rho_{max} = \frac{1}{e} \text{ at } G = 1$$

1) Suppose want to keep $G \approx 1$, then

$$prob\{idle\ \text{slot}\} = \text{e}^{\text{-G}} = e^{-1} \cong .368$$

$$prob\{successful\ transmission\} = \rho = e^{-1} \cong .368$$

$$prob\{collision\} = 1 - \frac{2}{e} \cong .264$$

So, if you know *n* exactly, can control *G(n)*, $G(n) \approx 1$

and achieve $\rho \approx \frac{1}{e}$ $\Rightarrow$ need an estimator of n

The above assume that all nodes use the same γ

2) We can get better throughput (i.e., $\rho_{\max} > \frac{1}{e}$) if each node keeps track of its own history of retransmissions and the feedback history (idle, success, collision) → Splitting Algorithms

“Pseudo Bayesian” Algorithm

Assumption: new and collided packets are assumed to be backlogged

If there are *n* packets (including new arrivals) at the beginning of a slot, the attempt rate $= n\gamma$

$$success \ \text{prob.} = n\gamma(1-\gamma)^{n-1}$$

But, don’t know n and needs to be estimated online based on the knowledge that

1) The previous slot is idle or a packet was successfully transmitted
2) There was a collision in the previous slot.

If have $\hat{n}$, we set $\gamma = \min\{1, \frac{1}{\hat{n}}\}$ so that $G(n) \cong 1$

How to get the estimate $\hat{n}$?

Suppose that the prior prob of the number of backlogged packets n At slot k is poisson with mean $\hat{n}_k$ (i.e. just before we know what happened In slot k)

$$p(n) = \frac{(\hat{n}_k)^n e^{-\hat{n}_k}}{n!}$$

Suppose slot k is idle, then

$$p(n \mid slot\ \text{k is idle}) = \frac{\text{p(slot k is idle} \mid \text{n)p(n)}}{\text{p(slot k is idle)}}$$

Since each node transmits with prob $\frac{1}{\hat{n}_k}$

$$prob\{slot\ \text{k idle} \mid \text{n}\} = (1 - \frac{1}{\hat{\text{n}}_\text{k}})^n$$

$$prob\{\text{slot k idle}\} = \sum_{\text{n}=0}^{\infty} (1 - \frac{1}{\hat{\text{n}}_\text{k}})^n \frac{e^{-\hat{n}_k}}{n!} (\hat{n}_k)^n = e^{-1}$$

So,

$$p(n \mid slot\ \text{k is idle}) = \frac{(1 - \frac{1}{\hat{\text{n}}_\text{k}})^n e^{-\hat{n}_k} (\hat{n}_k)^n}{n! e^{-1}} = \frac{e^{-(\hat{n}_k - 1)} (\hat{n}_k - 1)^n}{n!}$$

$$= \text{poisson with mean } \hat{\text{n}}_\text{k} - 1$$

$$\text{E(n} \mid \text{slot k is idle)} = \hat{\text{n}}_\text{k} - 1$$

Similarly,

$$p(n+1 \mid slot \text{ k is successful}) = \text{poisson with mean } \hat{\mathrm{n}}_{\mathrm{k}} \text{ -1}$$

$$\therefore \mathrm{E}(\mathrm{n} \mid \text{slot k is successful}) = \hat{\mathrm{n}}_{\mathrm{k}} \text{ -1}$$

$$\mathrm{p}(\mathrm{n} \mid \text{collision in slot k}) = \frac{\mathrm{p}(\text{collision in slot k} \mid \mathrm{n})\mathrm{p}(\mathrm{n})}{\mathrm{p}(\text{collision in slot k})}$$

$$= \frac{\mathrm{e}}{\mathrm{e}-2}\left[\frac{e^{-\hat{n}_k}(\hat{n}_k)^n}{n!} - \frac{e^{-\hat{n}_k}(\hat{n}_k-1)^n}{n!} - \frac{e^{-\hat{n}_k}(\hat{n}_k-1)^{n-1}}{(n-1)!}\right]$$

Not poisson, but assume it anyway

$$\therefore \mathrm{E}(\mathrm{n} \mid \text{collision in slot k}) = \frac{e}{e\text{-}2}\hat{n}_k - \frac{1}{e-2}(\hat{n}_k-1) - \frac{1}{e-2}\hat{n}_k$$

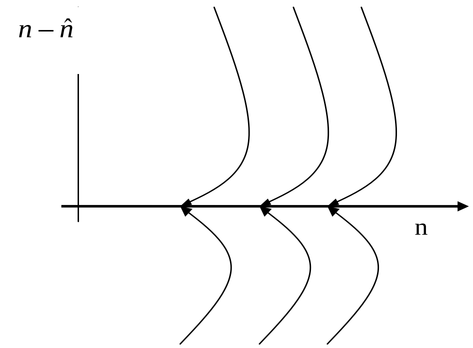
$$= \hat{n}_k + \frac{1}{e\text{-}2}$$

So, $$\hat{n}_{k+1} = \begin{cases} \max\{\rho, \hat{n}_k + \rho - 1 & \text{if idle success} \\ \hat{n}_k + \rho + \dfrac{1}{e-2} & \text{if collision} \end{cases}$$

Where, ρ accounts for new arrival during slot k.

Let us look at the stability of the system for $\lambda < \dfrac{1}{e}$

State of the system is $(n, \hat{n})\ or\ (\mathrm{n}, \mathrm{n} - \hat{\mathrm{n}})$



$$E(n_{k+1} - n_k) = \rho - \frac{1}{e} < 0$$

$$E(\hat{n}_{k+1} - \hat{n}_k) = \frac{2}{e}(\rho - 1) + (1 - \frac{2}{e})(\rho + \frac{1}{e-2})$$

$$= \frac{2}{e}(\rho - 1) + \frac{1}{e} + (e-2)\frac{\rho}{e} = \rho - \frac{1}{e} < 0$$

If $|n-\hat{n}|$ is large $\Rightarrow \gamma$ is too high or too low depending on whether $(n-\hat{n}) > 0$ or $(n-\hat{n}) < 0$

$(n-\hat{n}) >> 0 \Rightarrow$ You may decide to send at a prob more than necessary

$\Rightarrow$ more collisions $\Rightarrow n \uparrow$

$(n-\hat{n}) << 0 \Rightarrow$ You may decide not to send

$\Rightarrow$ *idle* states $\Rightarrow n \uparrow$ due to arrivals, but $E|n-\hat{n}| \downarrow$

Eventually $n \rightarrow \hat{n}$ and $n \downarrow$

Splitting algorithm:

All of them have some form of tree structure to resolve conflicts.

Suppose a collision occurs in slot k, then the collisions are resolved as follows:

- All nodes not involved in the collision go into a waiting node
- All nodes involved in the collision do the following
  - Split into two subsets (e.g. by flipping a coin)

    this splitting may also be based on time of arrival

    The first subset transmits in slot (k+1)

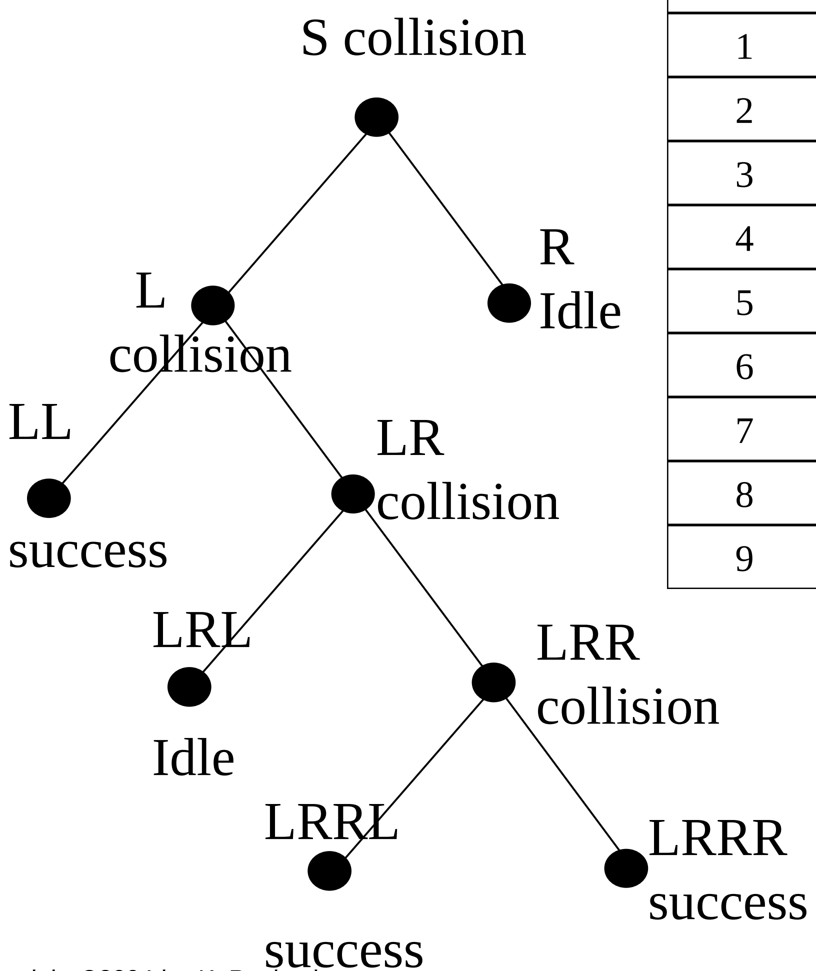
    If slot is idle or successful

    second subset transmits in slot (k+2)

    else (i.e., collision)

    split again and continue

3 packets



slot	transmit set	Waiting set	Feedback
1	S	---	E (error)
2	L	R	E
3	LL	LR, R	1 (success)
4	LR	R	E
5	LRL	LRR, R	0 (idle)
6	LRR	R	E
7	LRRL	LRRR, R	1
8	LRRR	R	1
9	R	---	0

- We can implement this algorithm using a stack.
- A node can keep track of when to transmit

$$\textit{packet} \text{ collided} \Rightarrow \text{set counter to } \overset{0 \text{ or } 1}{\text{L} \quad \text{R}}$$

$$\textit{counter} = 0 \quad \Rightarrow \text{transmit}$$

$$\text{counter} \neq 0 \quad \Rightarrow \text{counter} = \begin{cases} \text{counter} + 1; & \text{for collision} \\ \text{counter} - 1; & \text{for idle or success} \end{cases}$$

What to do with new packet?

- Wait until the collision resolution period (CRP) ends

CRP is large ⇒ lot of arrivals ⇒ larger CRP and so on.

Solution : split the nodes into j subsets, where *j* is chosen such that E{# of elements in a subset} is slightly greater than 1.

Place subsets in a stack and start the new CRP.

Capetanakis : Max throughput = 0.43 packets/slot

Improvement to the tree algorithm:

1) Collision followed by an idle slot $\Rightarrow$ one subset is null and the other subset is the complete subset. So, collision LRR is available.

To improve throughput,

omit transmission of second subset

split it into two subsets of

transmit the first of the split subsets

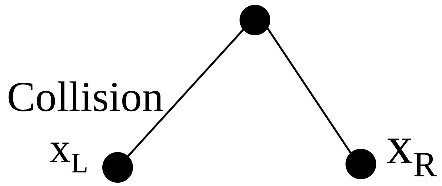
If on idle occurs, split the second subset again

This can be easily accomplished by each node by having an extra bit to keep track of idle slots following collisions.

This improves maximum throughput to 0.46.

2) Suppose have a collision followed by a collision

subset of nodes with x packets, collision $\Rightarrow x_L + x_R \geq 2$



$$x_L + x_R = x$$

Since equal split x_L and x_R are poisson if x is poisson

Collision implies $x_L \geq 2$

So,

$$P(x_k = i \mid x_L \geq 2, x_L + x_R \geq 2) = P(x_R\text{-}i \mid x_L \geq 2)$$

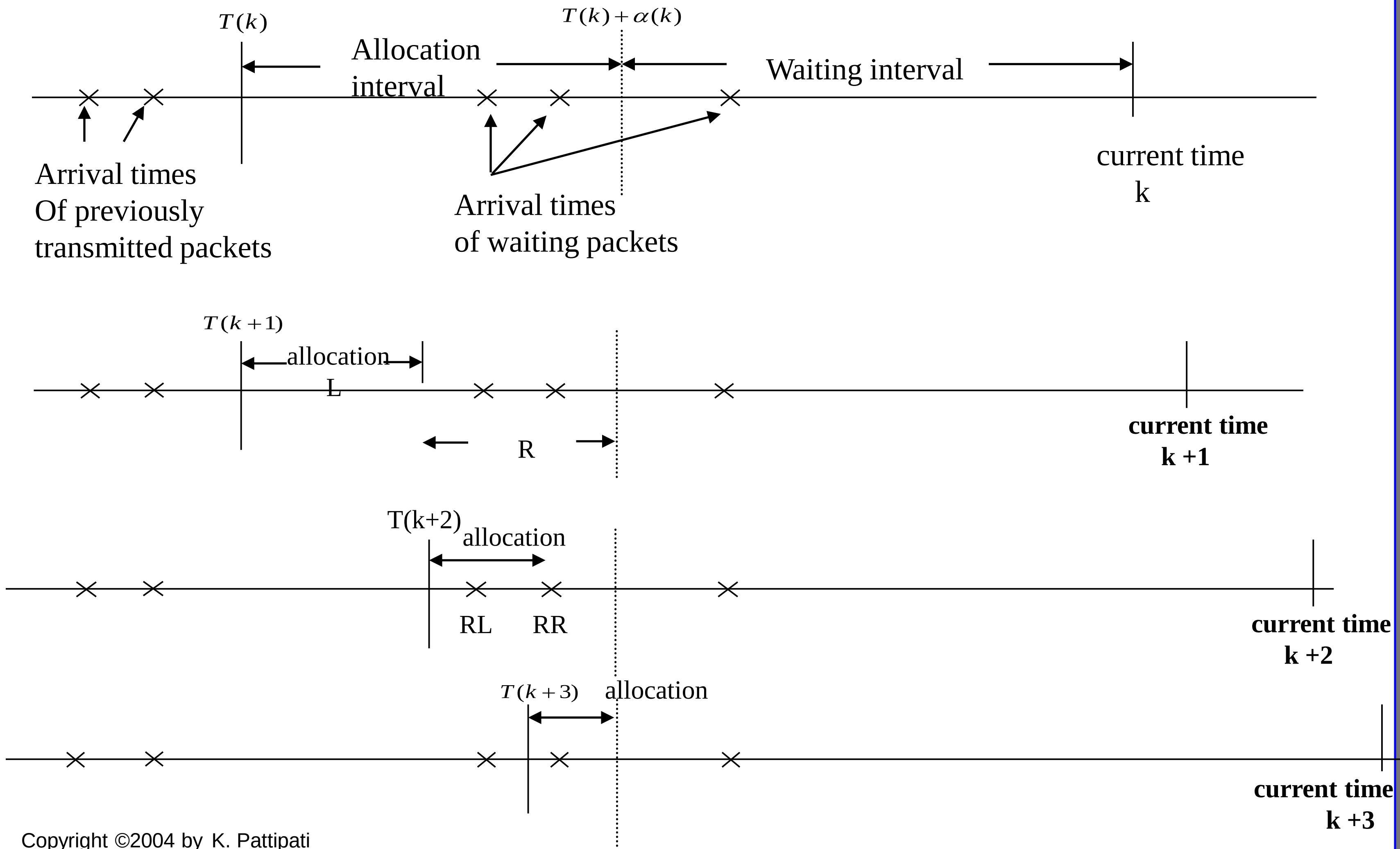
Since expected number of packets for x_R is "small", treat x_R as if they are new arrivals $\Rightarrow$ they are not part of current CRP

FCFS splitting algorithm

"Split the subset on the basis of arrival intervals"

At each time slot k, the algorithm specifies the packets to be transmitted to be the set of packets that arrived in some earlier interval $(T(k), T(k)+\alpha(k))$

$(T(k), \mathrm{T}(k) + \alpha(k))$ is the allocation interval



- *packets* arriving after $T(k)+\alpha(k)$ are in the queue (waiting).
- *packets* arriving during $[T(k), T(k)+\alpha(k)]$ are in service. But don't know the # of packets.

If collision split allocation interval into two parts and assign left most subinterval to the allocations subinterval to slot (k+1)

$$T(k+1) = T(k)$$
$$\alpha(k+1) = \frac{\alpha(k)}{2}$$

transmit packets in left subinterval

If success and was transmitting left subinterval packets

$$T(k+1) = T(k) + \alpha(k)$$
$$\alpha(k+1) = \alpha(k)$$

transmit packets in right subinterval

If idle and was transmitting left subinterval $\Rightarrow$ split right - most interval

$$T(k+1) = T(k) + \alpha(k)$$

$$\alpha(k+1) = \frac{\alpha(k)}{2}$$

Transmit left subinterval packets (i.e., left subinterval of split right subinterval)

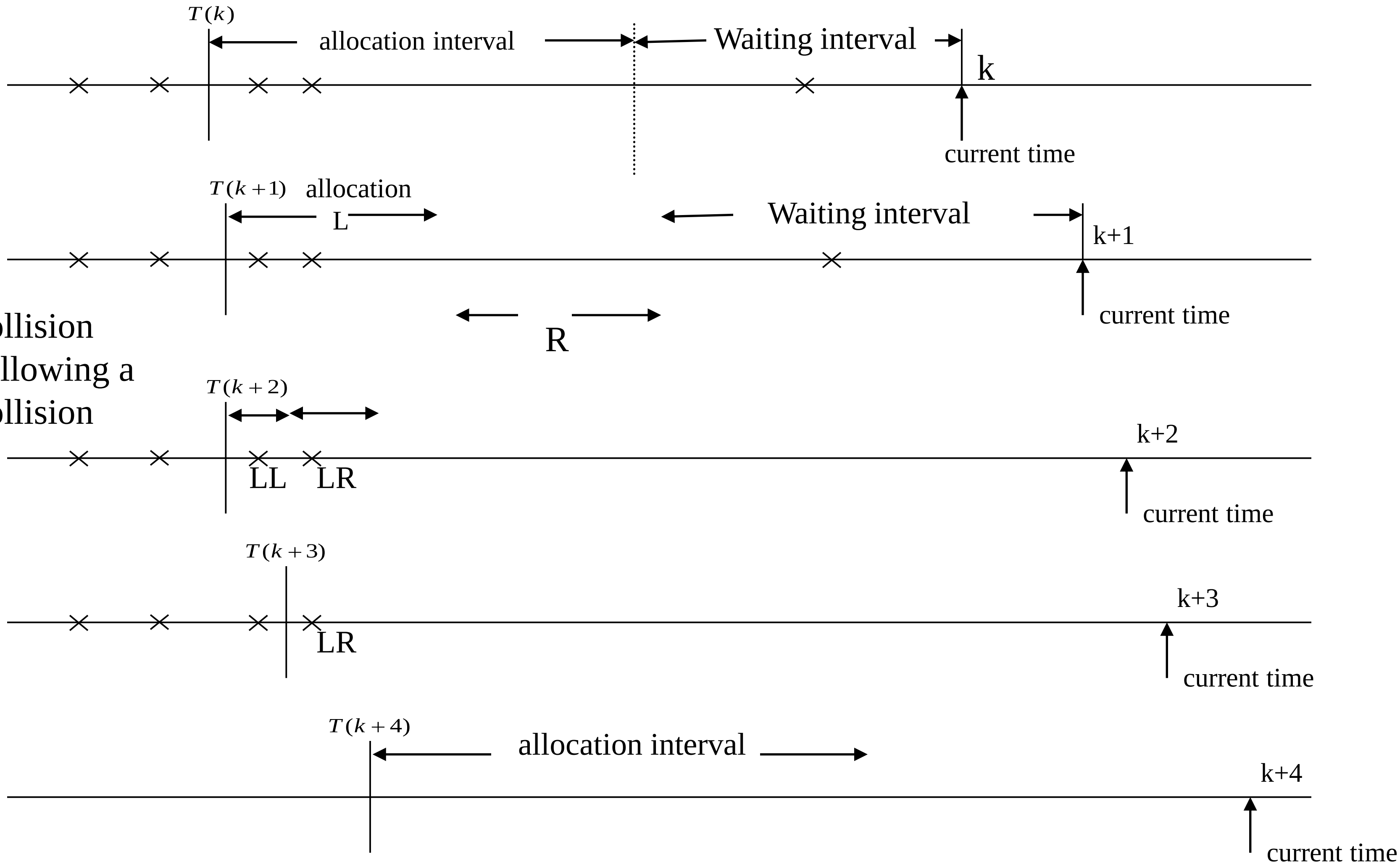
If idle or success and was transmitting right

$$T(k+1) = T(k) + \alpha(k)$$

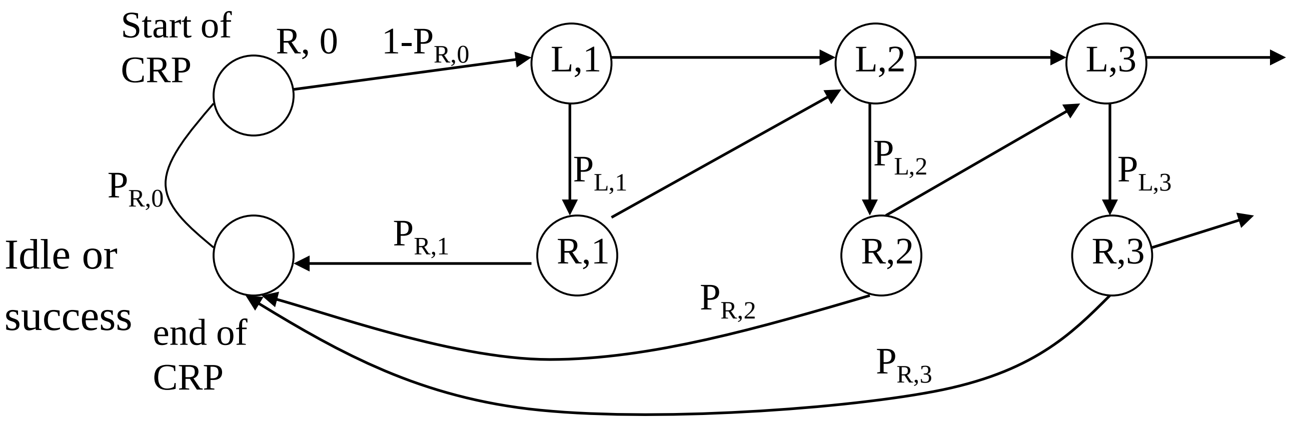
$$\alpha(k+1) = \min(\alpha_0, k+1-T(k+1))$$

Transmit packets in right subinterval

collision following a collision

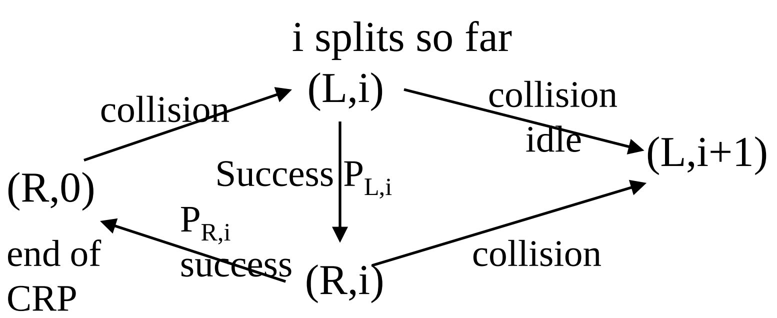


Markov Chain: Representation of Splitting Algorithm:



If a collision occurs (R,0) → (L,1)

left, one split



- Each split decreases the allocation interval by a factor of 2.

$$\text{i splits} \Rightarrow \alpha_0 \rightarrow 2^{-i}\alpha_0$$

- Average # of packets in the allocation interval $L_i = 2^{-i}\rho\alpha_0$

$$P_{R0} = prob\{\,idle \text{ or success}\}$$
$$= (1+\mathrm{L}_0)\mathrm{e}^{-\mathrm{L}_0}$$

- Transition from (L,1) → (R,1) occurs if packet is successfully transmitted

$$P_{L,1} = \frac{prob\{x_L = 1\}\,prob\{x_R \geq 1\}}{prob\{x_L + x_R \geq 2\}} = \frac{L_1 e^{-L_1}(1-e^{-L_1})}{[1-(1+L_0)e^{-L_0}]}$$

$$similarly,\ \mathrm{PR,1} = \frac{\mathrm{P}(\mathrm{x_e}=1)}{\mathrm{P}(\mathrm{x_e}\geq 1)} = \frac{L_1 e^{-L_1}}{1-e^{-L_1}}$$

In general,

$$P_{L,i} = \frac{L_i e^{-L_i}(1 - e^{-L_i})}{[1-(1+L_{i-1})e^{-L_{i-1}}]}$$

$$P_{R,i} = \frac{L_i e^{-L_i}}{1 - e^{-L_i}}$$

Prob of Markov states: $P(L,1) = 1 - P_{R,0}$

$$P(R,i) = P_{L,i}(L,i), \quad i \geq 1$$

$$P(L,i+1) = (1\text{-}P_{L,i})P(L,i) + (1\text{-}P_{R,i})P(R,i)$$

$$= P(L,i)\text{-}P_{L,i}(L,i) + P_{L,i}P(L,i)\text{-}P_{R,i}P_{L,i}P(L,i)$$

$$P(L,i+1) = (1 - P_{R,i}P_{L,i})P(L,i), \quad P(L,0) = 1, \; P_{LP} = 1$$

If we let E{k} be the average number of slots in a CRP, then

$$E\{K\} = 1 + \sum_{i=1}^{\infty}[P(L,i) + P(R,i)]$$

Note: $P(L,L+1)+P(R,i+1)=[1+PL,i(1-PR,i)]P(L,i)$

Re*call* that

$$1+P_{L,i}(1\text{-}P_{R,i})=\frac{1\text{-}(1+L^{i\text{-}1})e^{\text{-}L_{i\text{-}1}}+L_i e^{-L_i}[1-(1+L_i)e^{-L_i}]}{1\text{-}(1+L^{i\text{-}1})e^{\text{-}L_{i\text{-}1}}}$$

$$=1+L_i e^{\text{-}L_i}P\{collision\,|\,state(L,i)\}$$

- Change in T(k) from one CRP to the next

Initial allocation interval α_0

If left hand intervals have collisions, then the corresponding right hand Intervals are returned to the waiting interval.

If f is the fraction that is returned to waiting interval then change in $T(k)=\alpha_0(1-f)$

Prob of collision in state (L, i)

$=$ prob{left half interval} has at least two packets | right + left ≥ 2}

$$=\frac{1\text{-}(1+L_i)e^{\text{-}L_i}}{1\text{-}(1+L_{i\text{-}1})e^{\text{-}L_{i\text{-}1}}}=P(e|(L,i))$$

The fraction of the original interval returned on such a collision is 2^{-i}

So $$E(f) = \sum_{i=1}^{\infty} P(i,i) P(e \mid (L,i)) 2^{-i}$$

So, $E\{k\}$ *and* E{f} are f [ns] of L_i or of ρd_0

$$\textit{Drift}, \mathrm{D} = \mathrm{E}\{\mathrm{k} - \mathrm{T(k)}\} \text{ over a CRP}$$

$$\mathrm{D} = \mathrm{E}\{\mathrm{k}\} - \alpha_0[1 - E(f)]$$

$$D < 0 \ (\Rightarrow \text{stable}) \quad \text{if} \ \rho < \frac{\rho d_0 [1\text{-}E\{f\}]}{E\{k\}}$$

$$Max\ \rho = 0.4871 \text{ at } \rho\alpha_0 = 1.266 \ \text{(Numerical evaluation)}$$

$$\text{pick } \alpha_0 = 2.6$$

CS MA random access

- These are refinement on the pure and Slotted Aloha – We use additional hardware to detect (i.e., sense) the transmissions of other stations.

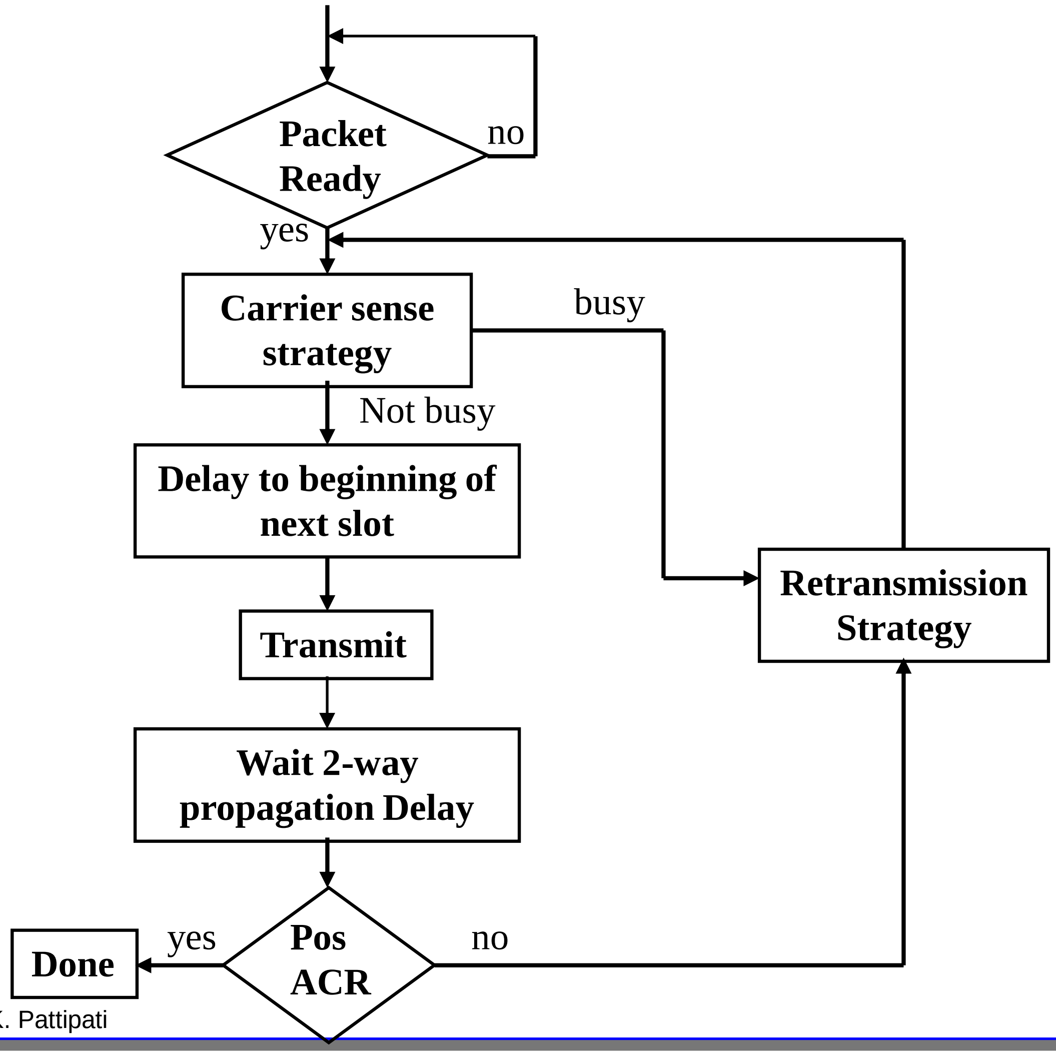
- Very useful for systems with propagation delays << packet transmission Times. Can have slotted or unslotted versions.

$$Let\ \tau = \text{propagation and detection delay to detect an idle channel after a transmission ends}$$

- $CSMA$ uses τ as the slot size

- If slotted, must tranmit at the beginning of a slot.

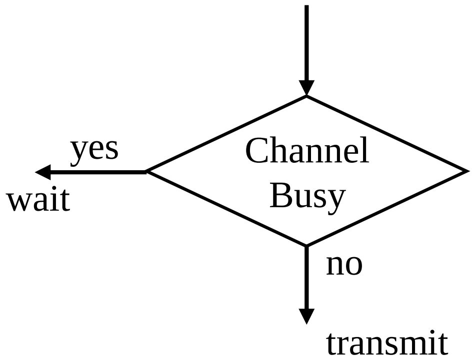
General: CSMA random access:



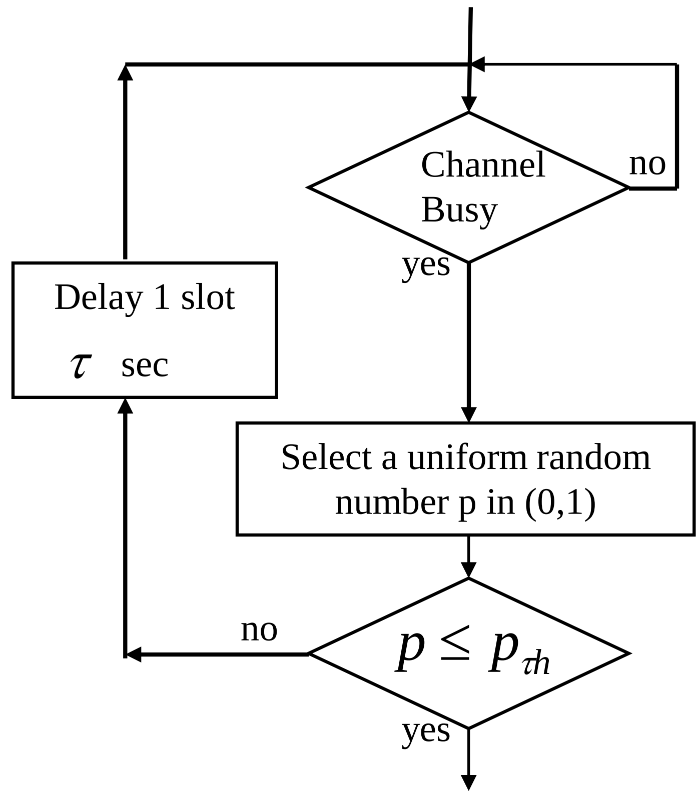
Two types of CSMA:

- Non-Persistent CSMA
- P- persistent CSMA

Non-persistent CSMA



p - persistent CSMA



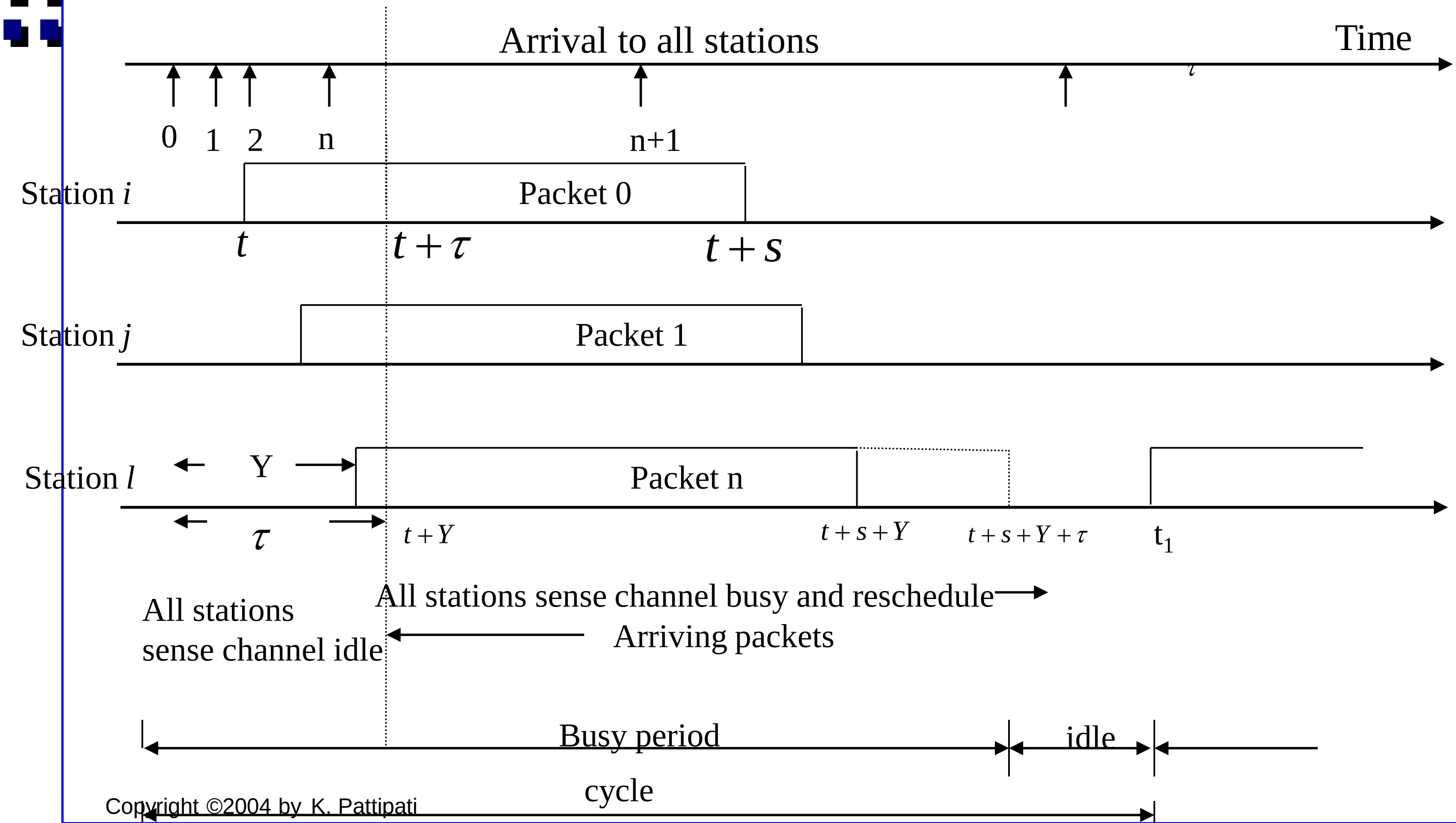
Analysis of unslotted CSMA random access procedures:

Model assumptions:

1. Number of users (nodes) is infinite and the arrival process is poisson.
2. Propagation and detection delay is τ seconds.
3. All packets have the same length and the same transmission time, *s*.
4. At any point in time, each node has at most one packet ready for transmission, including any previously collided packets.
5. Carrier sensing takes place immediately (instantaneous feedback)
6. Noise-free channel ➔ failure of transmission is due to collision only. Collision occurs whenever two packets overlap.

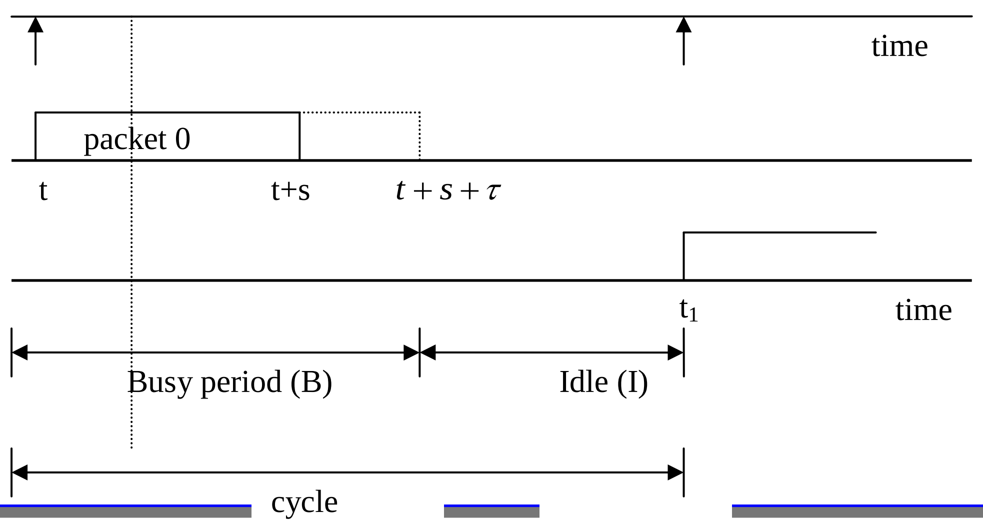
<u>Unsuccessful and successful busy periods for nonpersistent CSMA:</u>

<u>unsuccessful busy period</u>:



- Packet *o* arrives at the reference station at time *t*, since the channel is sensed idle, the packet is transmitted immediately.
- Packets 1,2,…, n do not know the existence of packet *o*. Let $t+Y$ be the time at which the last packet (in this case *n*) arrives before $t+\tau$.
- After $t+\tau$, stations know that channel is busy. So they reschedule packets for a later time (packet $n+1$).
- Packet *n* transmission ends by time $t+Y+s$ an all stations know about it by $t+Y+s+\tau$.

Successful busy period:



No arrival in $(t, t+\tau) \Rightarrow$ no collisions occur, y = 0

So, the vulnerable period for CSMA = propagation and detection delay
(Recall that for pure and slotted aloha it was *s* and *2s*, respectively.)

$$Throughput\ \ \rho = \frac{\text{Time over which useful work is performed}}{\text{cycle time}}$$

$$= \frac{\overline{U}}{\overline{B} + \overline{I}}$$

$\overline{U}$ = average time during a cycle where packets are successfully transmitted.

1) $\overline{U} = \text{s. prob}\{\text{packet } o \text{ is a good transmission}\}$

$= \text{s. prob}\{o \text{ arrivals in } (t, t+\tau)\}$

$= \text{s.e}^{-a\tau/s};$

$\frac{a}{s}$ = attempt rate (new + retransmissions), per packet transmission time

2) Busy period length

$$B = Y + S + \tau$$

know that

$$\mathrm{Y} = \begin{cases} 0 & \text{if successful transmission} \\ \text{some random variable, otherwise with prob } [1 - e^{-a\tau/s}] \end{cases}$$

Distribution of Y :

t + Y = time of last arrival of a packet in $(t, t+\tau)$

$\Rightarrow$ no packet in $(t + Y, t + \tau)$

$$F_Y(y) = prob\{Y \le y\} = prob\{no \text{ arrivals in } (t + y; t + \tau)\}$$

$$= e^{-a(\tau - y)/s} \quad \text{for } 0 \le y \le \tau$$

$$\bar{Y} = \int_0^{\tau} [1 - F_Y(y)]dy = \tau - \frac{s}{a}[1 - e^{-a\tau/s}]$$

$$so, \quad \overline{B} = 2\tau + s - \frac{s}{a}[1 - e^{-a\tau/s}]$$

as $a \to 0, \overline{B} \to \tau + s$ as it should

2) Mean Idle period, $\bar{I}$

$Inter-arrival$ times are exponential with mean $\frac{s}{a}$

$$so, \quad \bar{I} = \frac{s}{a}$$

$$so, \rho = \frac{\bar{U}}{\bar{B}+\bar{I}} = \frac{se^{-a\tau/s}}{s[1+\frac{2\tau}{s}]+e^{-a\tau/s}\frac{s}{a}}$$

If we let $\beta = \frac{\tau}{s}$ we have $\rho = \dfrac{ae^{-\beta G}}{a(1+2\beta)+e^{-\beta G}}$

For small β, $e^{-\beta G} \approx 1-\beta G$

$$\rho = \frac{e^{-\beta G}}{[\frac{1}{G}+(1+\beta)]}$$

The throughput has a maximum at

$$\frac{d\rho}{da} = 0 \Rightarrow -\beta e^{-\beta a}[\frac{1}{a} + (1+\beta)] + \frac{1}{a^2} e^{-\beta a} = 0$$

$$\text{or} \quad \frac{1}{\mathrm{a}^2} - \frac{\beta}{a} - \beta - \beta^2 = 0$$

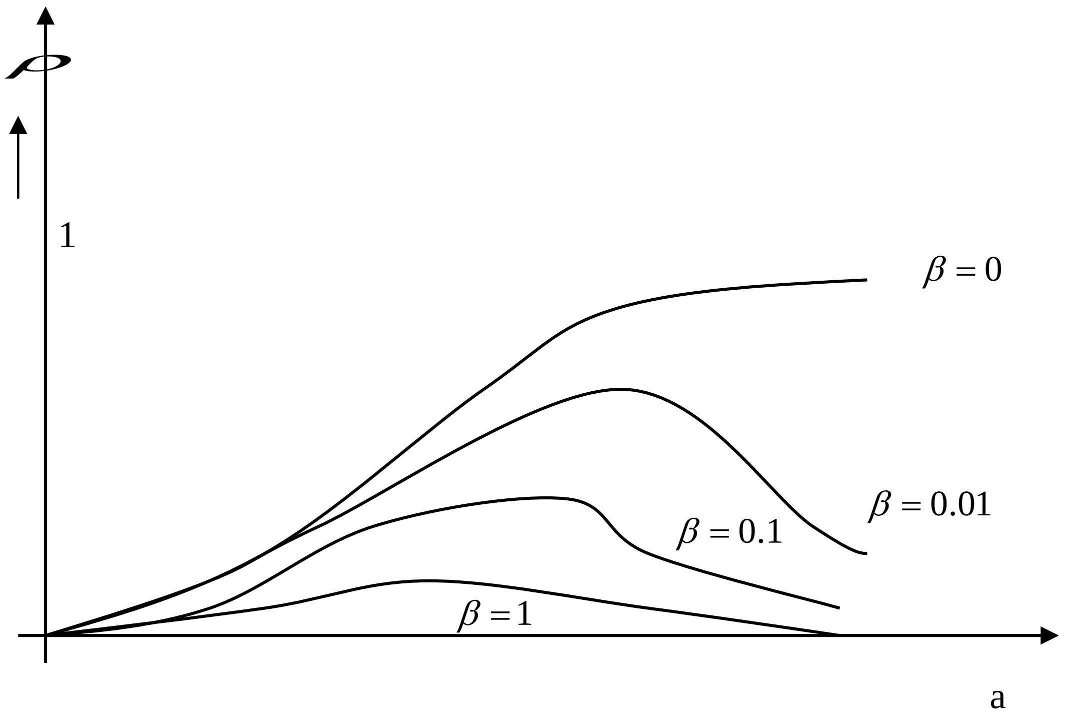
$$(\beta + \beta^2)\mathrm{a}^2 + \beta \mathrm{a} - 1 = 0$$

$$\mathrm{a}^2 + \mathrm{a} - \frac{1}{\beta} \approx 0$$

$$a = \frac{-1 + \sqrt{1 + 4/\beta}}{2} = \beta^{-1/2} - 1/2$$

$$\rho \max \cong \frac{1}{1 + 2\sqrt{\beta}}$$

$$\textit{Note}: \ \beta = 0 \Rightarrow \rho = \frac{\mathrm{a}}{1+\mathrm{a}} \quad \text{and } \rho_{\max} = 1$$



ρ
1
$\beta = 0$
$\beta = 0.01$
$\beta = 0.1$
$\beta = 1$
a

Typical behavior