Lecture 2

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ECE 336
Stochastic Models for the Analysis of Computer Systems and Communication Networks

Outline of Lecture 2

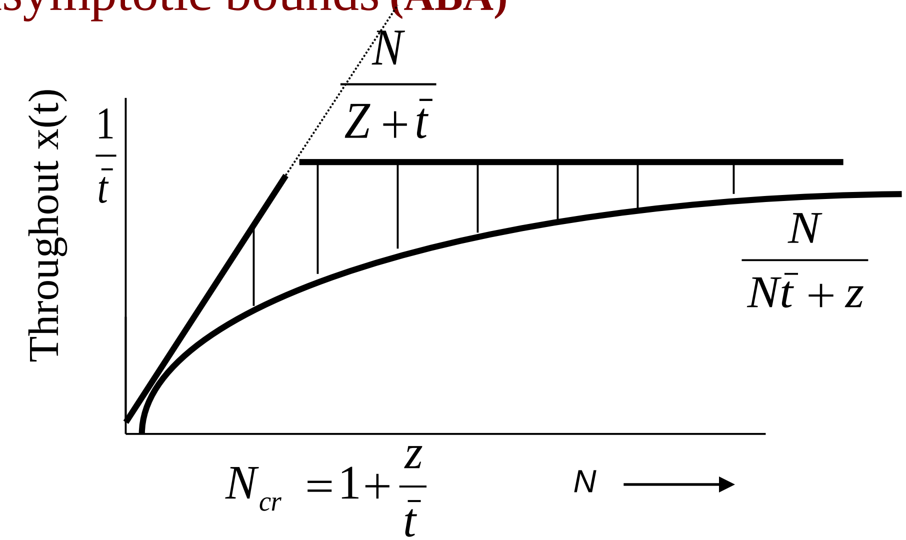
- Summary of Lecture 1
- Discrete-time Markov Chains
- Continuous-time Markov Chains
- Uniformization (Embedded Markov Chains)

Summary of Lecture 1

Characterizing Queuing Models : **Arrivals, Service, Queuing Discipline, Storage**

Little's law: $Q = \lambda R, \quad Q_w = \lambda W, \; U = \lambda \bar{t} \;\Rightarrow\; R = W + \bar{t}$

- When applied to a multi-access communication system, provided asymptotic bounds **(ABA)**



Can we get better information? yes, but requires the knowledge of the stochastic process *A(t)*, *D(t)* and *Q(t)* $\Rightarrow$ Need to have background in probability theory and stochastic processes

Random Variables

Examples:

- A Communication channel is (busy, idle)
- A Component is (up, down)
- A program is in one of n states (wait, execute, I/o, system up/down)

The possible observations or sample space is denoted by ***Ω***

Example: $\Omega = (\omega_1, \omega_2) = (\text{busy, idle})$

If we assign a variable $X(\omega, t)$ ∋

$$X(\omega, t) = \begin{cases} 1 & \text{if } \omega = \omega_1 \text{ at time t (busy)} \\ 0 & \text{if } \omega = \omega_2 \text{ at time t (idle)} \end{cases}$$

X(ω , t) is a discrete-state rv at time *t*

X(ω , t) is termed the random variable (rv) at time *t*, i.e., functions defined on the sample space Ω. We omit ω from the definition of a random variable from now on for simplicity.

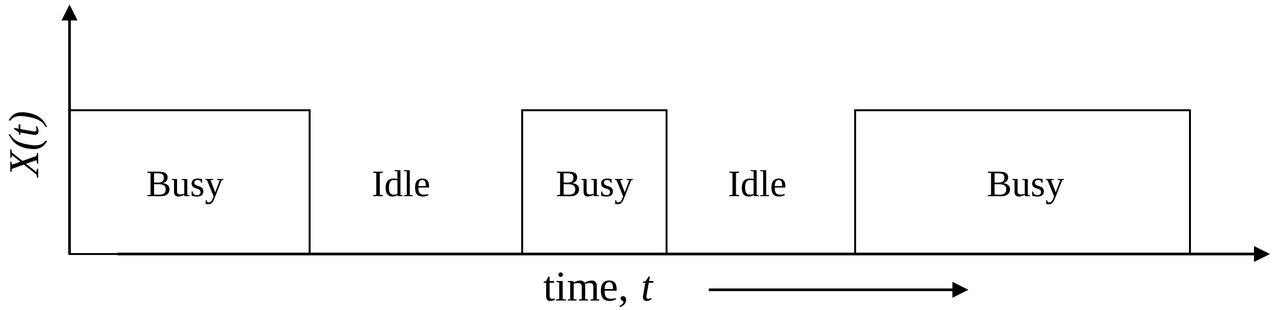
Stochastic Processes -1

Definition: A stochastic process is a family of random variables $\{X(t), t \in T\}$ where t varies over an index set T

- For a fixed value of index $t = t_1$, $X(t_1)$ is a random variable
- We can define four sets of stochastic processes depending on the possible values that X and t can take

Examples:

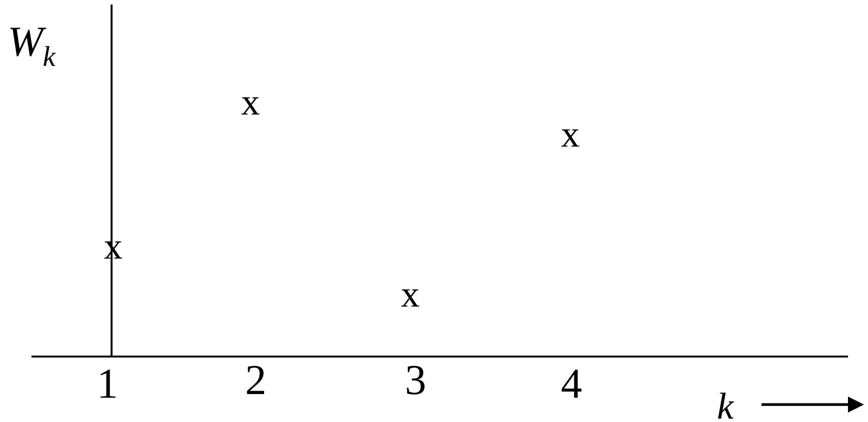
1) In the channel example above t is time; $T \in [0, \infty) \Rightarrow$ continuous-time so, the process is: Continuous-time Discrete-state (CTDS) process
   - A typical realization (or sample path) of the process consists of alternate busy and idle periods



- **$Q(t)$, $A(t)$, $D(t)$ are also CTDS processes**

Stochastic Processes -2

2) W_k = time k^{th} customer has to wait in the system before receiving service



$k \in T = \{ 1, 2, 3, 4 \ldots \} = Z^+$ set of positive integers > *0*

Discrete–index (Discrete-time)-continuous state (DTCS) process

- Discrete-time Control Systems

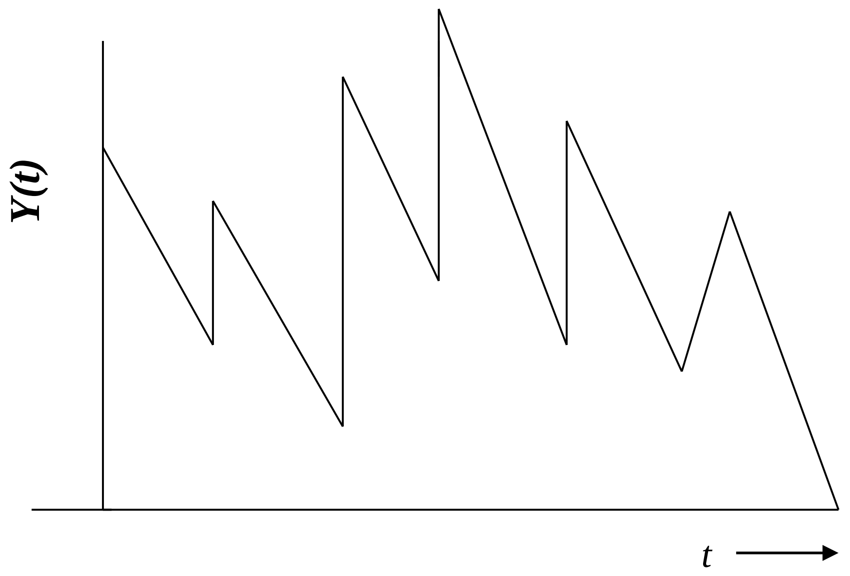
$$\left.\begin{aligned} \underline{x}_{k+1} &= \Phi \underline{x}_k + \Gamma \underline{u}_k + E \underline{w}_k \\ \underline{y}_k &= c \underline{x}_k + \underline{v}_k \end{aligned}\right\}$$

DTCS processes

Stochastic Processes - 3

3) Cumulative service demand of all jobs in the system ("workload")
$\{Y(t), t \in T \in [0, \infty)\}$

Continuous-time continuous-state (CTCS) process

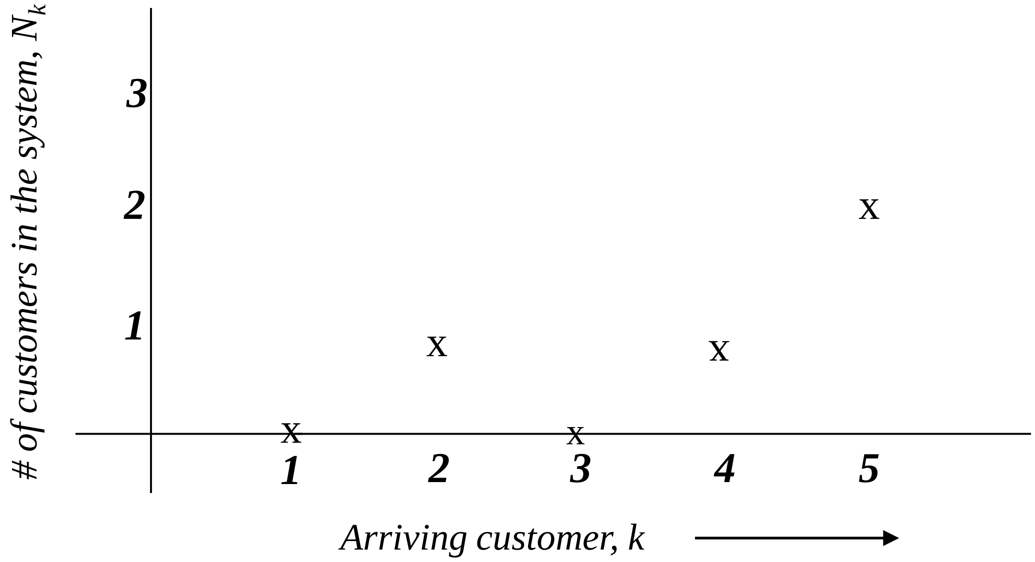


Stochastic Processes - 4

4) N_k = # of customers in the system at the time of arrival of k^{th} customer

$$= \{N_k \mid k = 1, 2, 3, \ldots\}, \quad N_k \in (0, 1, 2, 3, 4, \ldots)$$

Discrete – index (Discrete-time) Discrete – state (DTDS) stochastic process



Classification of Stochastic Processes

Classification Summary:

time \ state	Discrete	Continuous
Discrete	DTDS	DTCS
Continuous	CTDS	CTCS

DTCS… e.g., Delay analysis
DTDS… e.g., Markov Chains
CTDS… e.g., Continuous time Maikov chains

} Queuing & Reliability Applications

CTDS + CTCS… Performability processes

Probability Distribution

For a fixed time t_1, $X(t_1)$ is a random variable - For a random variable $X(t_1)$, we can talk about the cumulative distribution function (CDF)

$$F(x_1,t_1) = P\{X(t_1) \le x_1\}$$

Suppose we have sampled the process $X(t_1)$ at $t=t_1, t_2, \ldots, t_n$, then

$$F(\underline{x},\underline{t}) = P\{X(t_1) \le x_1;\ X(t_2) \le x_2; \cdots; X(t_n) \le x_n\}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \underline{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$$

Joint Distribution of $X(t_1), X(t_2) \ldots, X(t_n)$ is difficult to compute

We can make one of two assumptions.

Independent Process

- Independent process: Renewal process (e.g., failures of components with negligible repair time), X_i= inter-failure times i.i.d.

  By chain rule: $P(ABC) = P(A|BC)\, P(B|C)\, P(C)$

$$F(\underline{x},\underline{t}) = P\{\, X(t_n) \leq x_n \mid X(t_{n-1}) \leq x_{n-1}; \ldots\ldots; X(t_1) \leq x_1 \}\ldots$$

$$P\{X(t_2) \leq x_2 \mid \underline{x}(t_1) \leq x_1\},\ P\{X(t_1) \leq x_1\}$$

$$= \prod_{i=1}^{n} P(X(t_i) \leq x_i \mid X(t_1) \leq x_1, \ldots, X(t_{i-1}) \leq x_{i-1})$$

$$Independence \Rightarrow$$

$$F(\underline{x},\underline{t}) = \prod_{i=1}^{n} P(X(t_i) \leq x_i) = \prod_{i=1}^{n} F(x_i, t_i)$$

Markov Process

- Markov (first-order dependency) process

$$F(\underline{x},\underline{t}) = \prod_{i=1}^{n} P(\, X(t_i) \;\le\; x_i \mid X(t_{i-1}) \le x_{i-1})$$

More often then not, $X(t_{i-1})$ is known exactly, i.e. $X(t_{i-1}) = x_{i-1},$ so

$$F(\underline{x},\underline{t}) = \prod_{i=1}^{n} P(\, X(t_i) \;\le\; x_i \mid X(t_{i-1}) = x_{i-1})$$

The probability distribution of $X(t_i)$ at time t_i depends only on the state at $X(t_{i-1})$ time t_{i-1} for any sequence of time instants $t_1, t_2, \ldots, t_{i-1} \ni t_1 < t_2 < \ldots t_i$

"Knowledge of the present makes the past irrelevant"

Markov Chains

We are mostly interested in discrete-state Markov Process… known as Markov chains

Markov Chains

- Discrete-time (DTMC)
  - Transitions from one-state to the next are allowed at discrete time instants 0, Δ, 2Δ,…
- Continuous time (CTMC)
  - Transitions are allowed at any point in time

Similar to: $\underline{x}_{k+1} = P^T \underline{x}_k \qquad \dot{\underline{x}} = Q^T \underline{x} \;\Rightarrow \mathrm{P} = \mathrm{e}^{Q\Delta}$

For Markov chains, Q has a special structure:

$$Q\underline{e} = \underline{0} \ni \underline{e} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$\Rightarrow$ Eigen value $= 0$

$\Rightarrow$ Eigen vector $= \underline{e}$

$$P\underline{e} = \left(I + Q\Delta + \frac{Q^2\Delta^2}{2} + \cdots\right)\underline{e} = \underline{e}$$

$$\Rightarrow \textit{Eigen value} = 1 \;\&\; \textit{Eigen vector} = \underline{e}$$

Discrete-time Markov Chains -1

We will discuss some applications before we discuss the theory

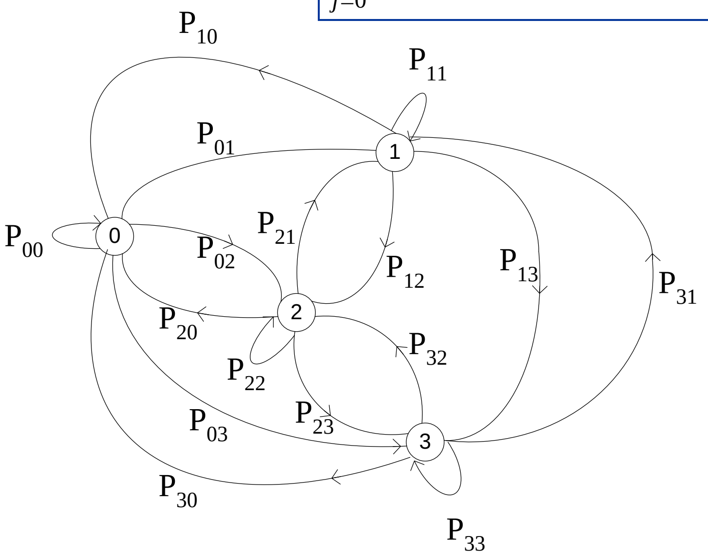
- DTMC: A process $\{X_n, n=0, 1, 2, \ldots\}$ is a finite-state Markov chain

$$\ni X_{n+1} \in (0, 1, 2, \ldots, N) \;\; and \;\; P(X_{n+1} = j | X_n = i) = P_{ij}(n) \geq 0$$

also

$$\sum_{j=0}^{N} P_{ij}(n) = 1 \quad \forall i \;\; \& \, n = 0, 1, 2, \ldots$$

$$\Rightarrow P\,\underline{e} = \underline{e}$$



State at step n+1 →

State at Step n ↓

	0	1	2	3
0	$P_{00}(n)$	$P_{01}(n)$	$P_{02}(n)$	$P_{03}(n)$
1	$P_{10}(n)$	$P_{11}(n)$	$P_{12}(n)$	$P_{13}(n)$
2	$P_{20}(n)$	$P_{21}(n)$	$P_{22}(n)$	$P_{23}(n)$
3	$P_{30}(n)$	$P_{31}(n)$	$P_{32}(n)$	$P_{33}(n)$

One-step Transition Probability Matrix (TPM)

$P_{ij}(n)$ ~ a function of step or stage n $\Rightarrow$ non-homogenous (non stationary or time-varying) Markov chain

P_{ij} = constant $\Rightarrow$ time homogenous (or stationary or time-invariant) Markov chain

Discrete-time Markov Chains -2

- If we let the unconditional probability that the chain is in state j at time step $(n+1)$ by $p_j(n+1)$ then

$$
\begin{aligned}
p_j(n+1) &\overset{\Delta}{=} P(X_{n+1}=j) \\
&= \sum_{i=0}^{N} P(X_{n+1}=j, X_n=i) \quad \text{by Total Prob. Theorem} \\
&= \sum_{i=0}^{N} P(X_{n+1}=j \mid X_n=i) \, . \, P(X_n=i) \\
&= \sum_{i=0}^{N} P_{ij}(n) p_i(n)
\end{aligned}
$$

$$\Rightarrow \boxed{\underline{p}(n+1) = P^T(n)\ \underline{p}(n)} \ ; \qquad p(n) = \begin{bmatrix} p_0(n) \\ p_1(n) \\ \vdots \\ p_N(n) \end{bmatrix}$$

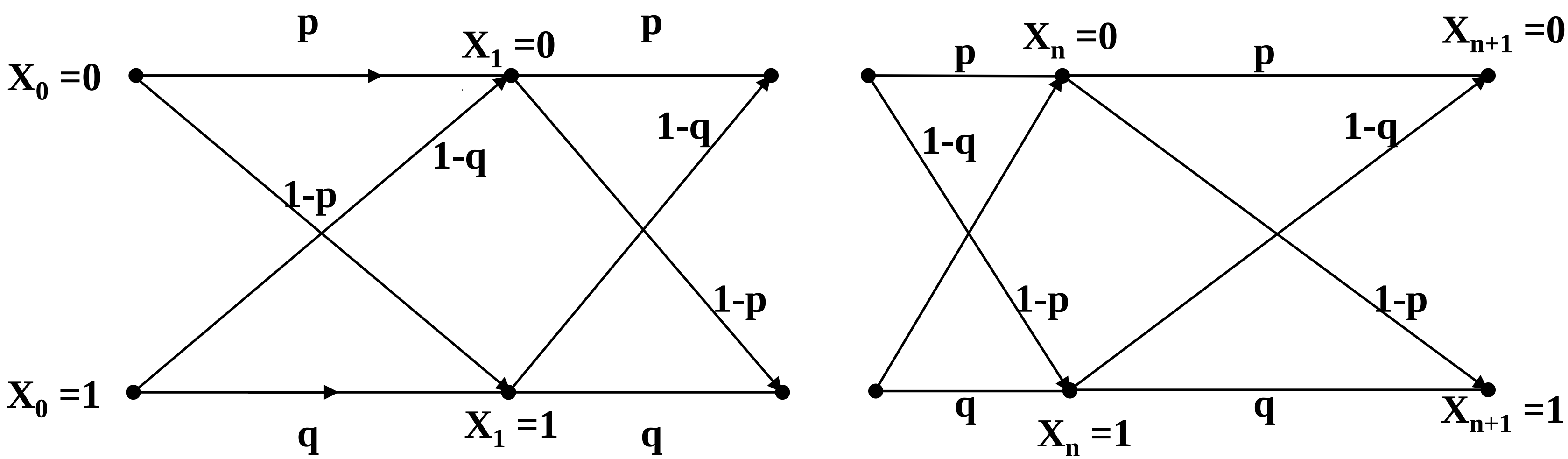
Non-homogeneous

$$\boxed{\underline{p}(n+1) = P^T\ \underline{p}(n)} \qquad \boxed{\underline{p}^T(n)\,\underline{e} = \sum_{i=0}^{N} p_i(n) = 1}$$

Homogeneous case

DTMC Example 1

Example: 1) A cascaded communication channel $X_n \in (0, 1)$



$$P(X_{n+1}=0/X_n=0)=p; \quad P(X_{n+1}=1\,|X_n=0) = 1\text{-}p$$
$$P(X_{n+1}=1/X_n=0)=1-q; \quad P(X_{n+1}=1|X_n=1) = q$$
$$P=\begin{bmatrix} p & 1\text{-}p \\ 1\text{-}q & q \end{bmatrix} \Rightarrow \lambda_1(p)=1,\, p+q-1$$

Eigen vectors:

$$\alpha\begin{bmatrix}1\\1\end{bmatrix};\alpha\begin{bmatrix}\frac{1-p}{q-p}\\ \frac{q-1}{q-p}\end{bmatrix}$$

Communication Channel Error Modeling

Notes:

1) The rows sum to unity $\|P\|_\infty = 1$

2) $P\,\underline{e} = \underline{e} \Rightarrow 1$ is an eigen value of P; $\underline{e}$ is an eigen vector of P

3) As $n \to \infty$, $\underline{p}(n) \to \underline{p}$

$$\Rightarrow \underline{p} = P^T \underline{p} \qquad \underline{p} \sim \left\{ \begin{array}{l} \text{Limiting probablity vector} \\ \text{Steady-state} \\ \text{stationary} \end{array} \right\}$$

When does it exist?

4) $Rank\,(P) \le N \Rightarrow (N+1)$ unknowns, at most N independent equations

$(N+1)^{th}$ equation: $\sum_{i=0}^{N} p_i = 1 \Rightarrow$ normalization equation

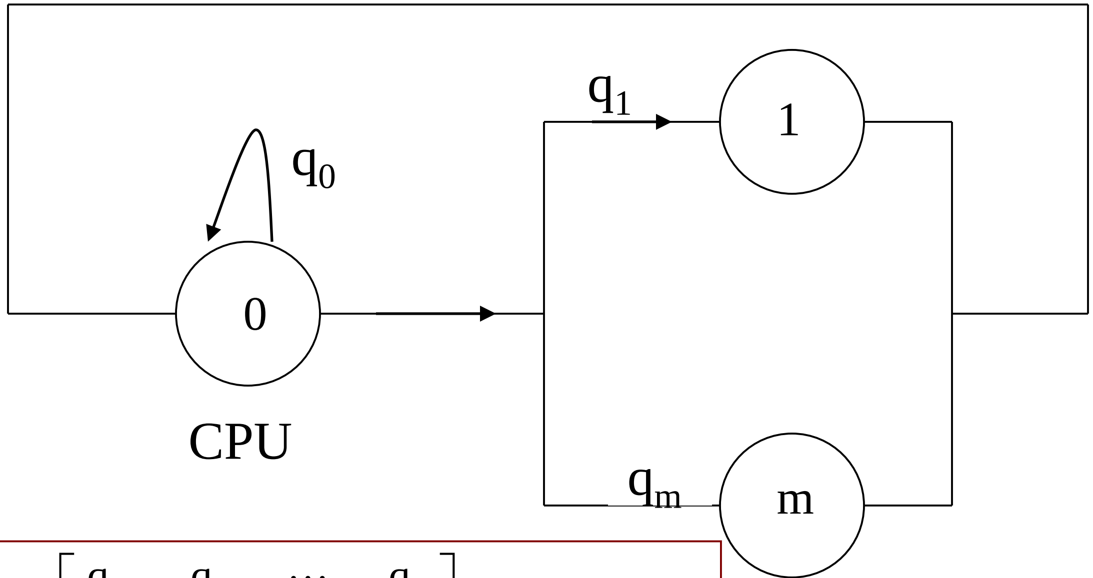
Steady state probabilities for comm. channel:

$$p_0 + p_1 = 1;\ pp_0 + (1-q)p_1 = p_0$$

$$\Rightarrow (p-1)p_0 + (1-q)(1-p_0) = 0\} \Rightarrow p_0 = \frac{1-q}{2-p-q};\ p_1 = \frac{1-p}{2-p-q}$$

DTMC Example 2

Example 2: Routing in a central server modal of a CPU-I/O subsystem
…..Uni-programmed (single job) system



Devices: i= 0, 1, 2, , m
X = state of job
$X= i$ if job is at device i
$\Rightarrow N=m$

Other Applications:

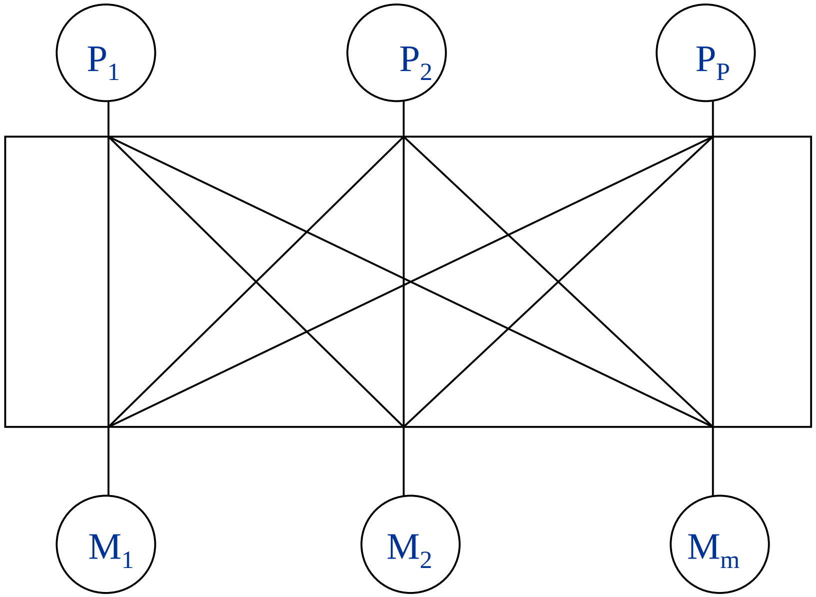
- Relative visits in closed networks
- Modeling programs

$$P = \begin{bmatrix} q_0 & q_1 & \cdots & q_m \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \Rightarrow \sum_{i=0}^{m} q_i = 1$$

$$\left. \begin{aligned} p_0 &= q_0\, p_0 + \sum_{i=1}^{m} p_i \\ p_i &= q_i\, p_0 \\ \sum_{i=0}^{m} p_i &= 1 \end{aligned} \right\} \Rightarrow \begin{aligned} p_0 &= \frac{1}{2-q_0} \\ p_i &= \frac{q_i}{2\text{-}q_0} \end{aligned}$$

DTMC Example 3

Example 3: Memory interference in a shared-memory multi-processor system



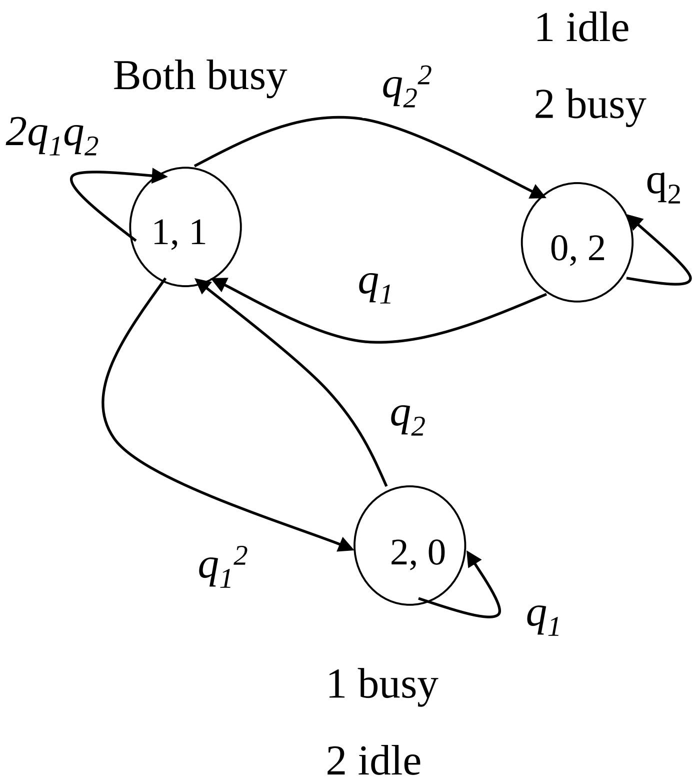
Assumptions

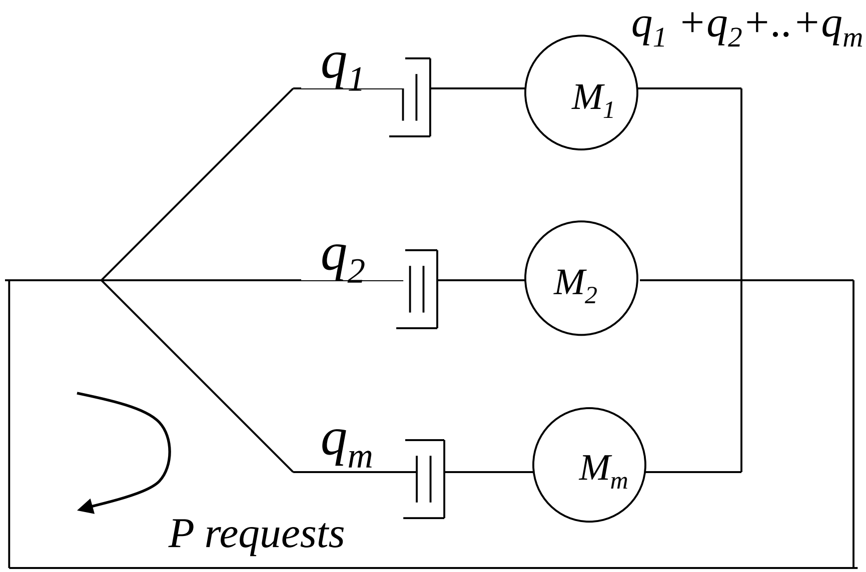
1. Cross – bar switch (m x P)
2. Memory split into m modules
3. Only one processor is granted access to a memory module
4. Processes generate new requests as soon as current request is processed $\Rightarrow$ P requests at memory modules.
5. Memory access time is a constant (= 1 unit of time) (deterministic)
6. q_i = prob. that a processor generates request to memory module i

Problem: Find average # of memory requests completed per memory cycle

Memory Interference in a Multi-processor

Consider $P=2$, $m=2$
Possible states: $(1, 1)$, $(0, 2)$, $(2, 0)$





$$P = \begin{array}{c} \\ 1\ 1 \\ 0\ 2 \\ 2\ 0 \end{array} \begin{array}{c} \begin{array}{ccc} 1\ 1 & 0\ 2 & 2\ 0 \end{array} \\ \begin{bmatrix} 2q_1q_2 & q_2^2 & q_1^2 \\ q_1 & q_2 & 0 \\ q_2 & 0 & q_1 \end{bmatrix} \end{array}$$

Note:
$(q_1 + q_2)^2 = 1$

Memory Interference in a Multi-processor

In Steady State

$$p_{(1,1)} = 2\, q_1\, q_2\, p_{(1,1)} + q_1 p_{(0,2)} + q_2 p_{(2,0)}$$

$$p_{(0,2)} = q_2^2 p_{(1,1)} + q_2 p_{(0,2)} \quad \Rightarrow \quad p_{(0,2)} = \frac{q_2^2}{1-q_2}\, p_{(1,1)}$$

$$p_{(2,0)} = q_1^2 p_{(1,1)} + q_1 p_{(2,0)} \quad \Rightarrow \quad p_{(2,0)} = \frac{q_1^2}{1-q_1}\, p_{(1,1)}$$

$$p_{(1,1)} = \frac{1}{1+\dfrac{q_1^2}{1-q_1}+\dfrac{q_2^2}{1-q_2}} = \frac{(1-q_1)(1-q_2)}{q_1^2+q_2^2-q_1\, q_2} = \frac{q_1\, q_2}{1-2\, q_1\, q_2}$$

We can compute other measure of interest. For example, the expected number of memory requests completed per memory cycle, *E(B)* can be computed as follows:

$$E(B) = E(B|[1,1])\, p_{(1,1)} + E(B|[0,2])\, p_{(0,2)} + E(B|[2,0])\, p_{(2,0)}$$

$$E(B\,|\,[1,1]) == 2;\; E(B|[0,2]) = 1;\; E(B|[2,0]) = 1$$

Memory Interference in a Multi-processor

$$\therefore E(B) = \left(2 + \frac{q_1^2}{1-q_1} + \frac{q_2^2}{1-q_2}\right) \frac{q_1 q_2}{1-2q_1q_2} = \frac{1-q_1q_2}{1-2q_1q_2}$$

- Optimization Problem:

$$\max_{q_1,q_2} E(B) \text{ subject to } q_1 + q_2 = 1 \Rightarrow q_1^* = q_2^* = \frac{1}{2} \Rightarrow E(B) = \frac{3}{2}$$

References:

1. F. Baskett and A.J. Smith, "Interface in Multi-processor Computer Systems with Interleaved Memory," CACM, Vol.19, No.6, 327-334, 1976.
2. D. Chang, D.J. Kuck, and D.H. Lawrie," On the Effective Bandwidth of Parallel Memories," IEEE Trans. on Computers, Vol C-26-5, May 1977, pp.480-42
3. S.H. Fuller, "Performance Evaluation," in Introduction to Computer Architectures, H.S. Stone (ed.), Science Research Associates, Chicago, IL, 1975.

DTMC ↔ CTMC - 1

Before considering some interesting properties of DTMC, let us introduce the corresponding continuous-time Markov chains (CTMC) so that we can study their properties by analogy.

DTMC ↔ CTMC

$$0 \quad \Delta \quad 2\Delta \quad \cdots \quad n\Delta \quad (n+1)\Delta \qquad \underset{\displaystyle n\Delta}{\overset{t}{\updownarrow}} \qquad \underset{\displaystyle (n+1)\Delta}{\overset{t+\Delta t}{\updownarrow}}$$

Know

$$\underline{p}((n+1)\Delta) = P^T(n\Delta)\, \underline{p}(n\Delta)$$

$$\underline{p}(t+\Delta t) = P^T(t)\, \underline{p}(t)$$

$$\underline{p}(t) = \begin{bmatrix} p_0(t) \\ p_1(t) \\ \vdots \\ p_N(t) \end{bmatrix}; \qquad p_i(t) = P\{X(t) = i\}$$

$$\underline{p}(t+\Delta t) - \underline{p}(t) = (P^T(t) - I)\, \underline{p}(t)$$

DTMC ↔ CTMC - 2

$$\lim_{\Delta t \to 0} \frac{\underline{p}(t+\Delta t) - \underline{p}(t)}{\Delta t} = \left(\lim_{\Delta t \to 0} \frac{P^T(t) - I}{\Delta t} \right) \underline{p}(t)$$

$$\frac{d\,\underline{p}(t)}{dt} = Q^T(t)\,\underline{p}(t)$$

where

$$Q(t) = \lim_{\Delta t \to 0} \frac{P(t) - I}{\Delta t} \Rightarrow q_{ii}(t) = \lim_{\Delta t \to 0} \frac{P_{ii}(t) - 1}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[P\{X(t+\Delta t) = i \mid X(t) = i\} - 1]}{\Delta t}$$

$$q_{ij}(t) = \lim_{\Delta t \to 0} \frac{P_{ij}(t)}{\Delta t}; \quad i \neq j$$

$Q(t) = [\, q_{ij}(t) \,]$ $(N+1)$ by $(N+1)$ matrix is termed the "infinitesimal generator matrix" or "the transition rate matrix"

DTMC ↔ CTMC - 3

Notes :

1) $P = e^{Q\Delta}$

2) Since $\sum_{j=0}^{N} P_{ij}(t) = 1 \Rightarrow \sum_{j=0}^{N} q_{ij}(t) = 0 \quad \forall i \Rightarrow$ row sums of Q are zero

$P\underline{e} = \underline{e} \Rightarrow e^{Q\Delta}\underline{e} = \underline{e} \Rightarrow Q\underline{e} = \underline{0}$

2) $q_{ij}(t) = q_{ij} \Rightarrow$ Homogenous Markov chain

3) $Q\underline{e} = \underline{0} \Rightarrow \lambda = 0$ is an eigen value of Q with eigen vector $\underline{e}$

4) Steady state probability distribution :

CTMC ↔ DTMC

$$\dot{\underline{p}} = 0 \Rightarrow Q^T \underline{p} = \underline{0} \Leftrightarrow \underline{p} = P^T \underline{p}$$

Since Rank $(Q) \leq N \Rightarrow$ at most N independent equations

$(N+1)^{th}$ equation $\sum_{i=0}^{N} p_i = 1$

$\underline{p}$ = normalized (with 1-norm) eigen vector of Q^T for eigen value = 0

or normalized eigen vector of P^T for eigen value 1.

When does it exist?

What do Transition Rates Mean?

What do the transition rates means ?

- Given that the process is in state i at time t, then the probability that a transition occurs to any other state during the interval $(t, t+\Delta t]$ is given by $-q_{ii}(t)\,\Delta t + o(\Delta t) \Rightarrow -q_{ii}(t)$ is the rate at which the stochastic process leaves state i at time t, given that the process is in state i at time t

$$q_{ii}(t) = -\sum_{i \neq j} q_{ij}(t) \;\; (or) \;\; 1 = -\sum_{i \neq j} \frac{q_{ij}(t)}{q_{ii}(t)} = \sum_{i \neq j} \frac{q_{ij}(t)}{\lambda_i};\; \lambda_i(t) = -q_{ii}(t)$$

- Given that the process in in sate i at time t, the conditional probability that it will make a transition to state j in the time interval $(t, t+\Delta t]$ is given by

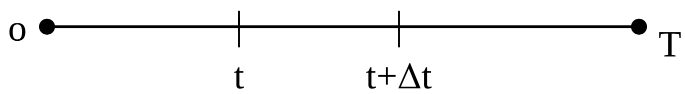
$$q_{ij}(t)\,\Delta t + o(\Delta t) \Rightarrow P\{X(t+\Delta t) = j \mid X(t) = i\} = q_{ij}(t)\,\Delta t + o(\Delta t);\; i \neq j$$

$\Rightarrow q_{ij}(t)$ is the rate at which the process moves from state i to state j at time $t+\Delta t$, given that the system is in state i at t

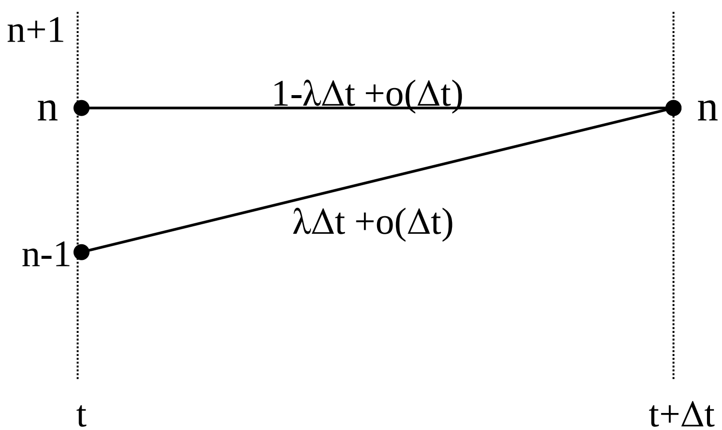
CTMC Example 1

- Example: Poisson process … simplest form of continuous-time Markov chain …. also know as pure-birth process

Suppose we observe the arrival of messages at a communication channel (or # of failures or jobs at a computer center) for the time interval $(0, T)$. Let $X(t)$ denote the number of messages (or jobs) at time t. $X(0) = 0 \Rightarrow P\{X(0) = 0\} = p_0(0) = 1$



For Δt "small", we assume that



$$P\{X(t+\Delta t) = n\} \stackrel{\Delta}{=} p_n(t+\Delta t)$$
$$= p_{n-1}(t)\,\lambda\,\Delta t + p_n(t)\,(1-\lambda\,\Delta t)$$
$$\Rightarrow \frac{dp_n(t)}{dt} = \lambda\; p_{n-1}(t) - \lambda\; p_n(t);\; n \geq 1;\; p_0(0) = 1$$

$$\begin{bmatrix} \dot{p}_0 \\ \dot{p}_1 \\ \dot{p}_2 \\ \vdots \\ \dot{p}_n \\ \vdots \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \lambda & -\lambda & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \lambda & -\lambda & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \lambda & -\lambda & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ \vdots \\ \vdots \\ p_n \end{bmatrix} \Rightarrow \begin{matrix} q_{ii} = \lambda \\ q_{ii+1} = \lambda \end{matrix}$$

$$\dot{p}_o(t) = -\lambda\, p_o(t)$$
$$\dot{p}_n(t) = \lambda\, p_{n-1}(t) - \lambda\, p_n(t);\; n \geq 1$$

Differential – difference equation

Digression : Moment Generating Function

- Moment Generating Function (MGF): suppose a discrete random variable *X(t)* assumes values 0, 1, 2, … with probability, $p_n(t)\,;\, n \ge 0$

$$G_{X(t)}(z,t) = E[z^{X(t)}] = \sum_{n=0}^{\infty} p_n(t)\, z^n = p_0(t) + p_1(t) z + \cdots$$

$$G_{X(t)}(1,t) = \sum_{n=0}^{\infty} p_n(t) = 1$$

$$\frac{\partial G_{X(t)}(z,t)}{\partial z} = \sum_{n=0}^{\infty} n\, p_n(t)\, z^{n-1}\,;\quad \frac{\partial G_{X(t)}(1,t)}{\partial z} = E[X(t)]$$

$$\frac{\partial^2 G_{X(t)}(z,t)}{\partial z^2} = \sum_{n=0}^{\infty} n\,(n\text{-}1)\, p_n(t)\, z^{n-2} \Rightarrow \frac{\partial^2 G_{X(t)}(1,t)}{\partial z^2} = E[X^2(t)] - E[X(t)]$$

$$E[X^2(t)] = \frac{\partial^2 G_{X(t)}(z,t)}{\partial z^2}\Big|_{z=1} + E[X(t)]$$

$$\sigma^2_{X(t)} = E[X^2(t)] - \{E[X(t)]\}^2$$

$$C_x = \frac{\sigma_{X(t)}}{E[X(t)]} \Rightarrow \textit{coefficient of variation};\ C_x \uparrow \ \Rightarrow\ \textit{larger variability}$$

MGF provides a simple method of evaluating moments of random variables

Poisson Process

❑ Coming back to Poisson process

$$\sum_{n=0}^{\infty}\frac{dp_n(t)}{\partial t}z^n = \lambda\sum_{n=0}^{\infty}p_{n-1}(t)z^n - \lambda\sum_{n=0}^{\infty}p_n(t)z^n \;;\quad \frac{\partial G_{X(t)}(z,t)}{\partial z}\Big|_{z=1} = E[X(t)]$$

$$\frac{\partial G(z,t)}{\partial t} = \lambda z\, G(z,t) - \lambda\, G(z,t) = \lambda(z-1)G(z,t)$$

$$\Rightarrow G(z,t) = e^{\lambda t(z-1)}G(z,0) = e^{\lambda t(z-1)} = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}z^n$$

$$Since \quad p_0(0)=1 \;\;\&\;\; p_n(0) = 0 \;\; for \;\; n\geq 1 \Rightarrow G(z,0) = 1$$

$$\therefore\; p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}\;;\; n = 0, 1, 2 \cdots$$

- For $\lambda t>30$, Poisson ~ Normal
- Square root of $X(t)$ is nearly normal with variance 0.25
- More properties in Lecture 3

❑ Moments of Poisson process

$$E[X(t)] = \lambda t\, e^{-\lambda t(z-1)}\big|_{z=1} = \lambda t$$

$$E[X^2(t)] = (\lambda t)^2 + \lambda t \Rightarrow \sigma^2_{X(t)} = \lambda t = mean$$

$$C_{X(t)} = \frac{1}{\sqrt{\lambda t}} \Rightarrow C_{X(t)} \to 0 \text{ as } t\to\infty \Rightarrow \text{impulse at the mean } \lambda t$$

$$\text{so, } \lambda = \lim_{t\to 0}\frac{\#\text{ of arrivals (or events) in } (0,t)}{t} = \text{"rate of arrivals"}$$

Poisson Process : Three Definitions

- Counting process $\{X(t), t \geq 0\}$ is Poisson with rate $\lambda > 0$, if

$(i)\ N(0) = 0$

(ii) has independent increments, and

(iii) # of events in any interval of length t is distributed as

$$p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}; n = 0,1,2,..$$

$\longleftrightarrow$

$(i)\ N(0) = 0$

(ii) has stationary & independent increments, and

$(iii)\ P(X(\Delta t) = 1) = \lambda \Delta t + O(\Delta t)$

$(iv)\ P(X(\Delta t) \geq 2) = O(\Delta t)$

The inter-arrival times $\{\tau_n; n \geq 1\}$ *are i.i.d. exponential random variables having mean* $1/\lambda$

$$\overline{H}_{\tau_i}(t) = P(\tau_i > t) = P(X(t) = 0) = e^{-\lambda t}$$

$$h_{\tau_i}(t) = \lambda e^{-\lambda t}; t \geq 0$$

Distribution of Time between State Changes

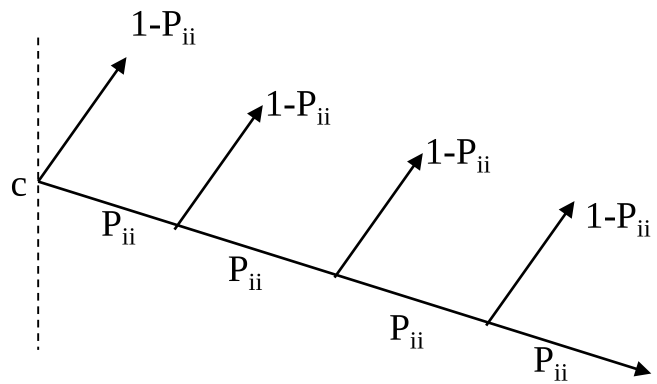
- Distribution of time between state changes in a DTMC and CTMC

  DTMC ↔ Geometric pmf

  CTMC ↔ Exponential density

  Homogeneous case

- DTMC: Suppose that the Markov chain is in state i at time $0 \Rightarrow X(0) = i$
  - The chain will remain in state i with probability P_{ii} and it will leave the state with probability $(1- P_{ii})$
  - Suppose that the next state is i ($\Rightarrow X(1) = i$). Then, the same two choices are available at the next time step



Let T_i = time the Markov chain spends in state i during a single visit to state i
= 1+ # of ($i \rightarrow i$) transitions made before leaving state i

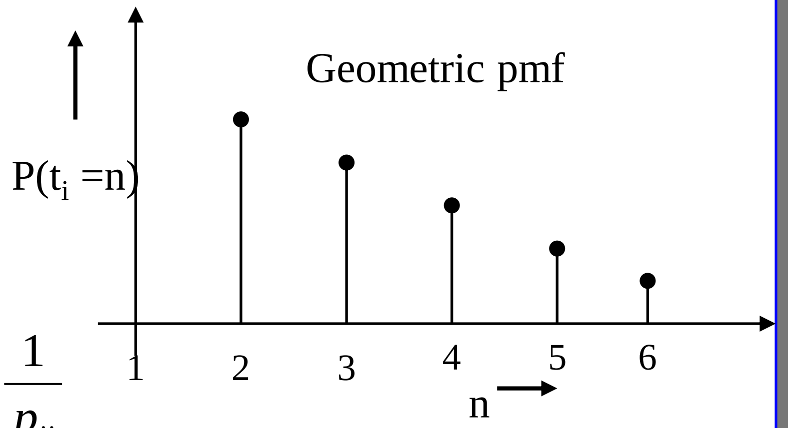
$$T_i = 1 \text{ with probability } (1\text{-}P_{ii})$$
$$T_i = 2 \text{ with probability } (1\text{-}P_{ii})\, P_{ii}$$
$$T_i = n \text{ with probability } (1\text{-}P_{ii})\, P_{ii}^{n-1}$$

Geometric pmf

Geometric pmf

- Geometric pmf

$$P(T_i = n) = (1 - P_{ii})\, P_{ii}^{n-1}\ ; n = 1, 2, \ldots$$



- Moments of Geometric pmf

$$G(z) = \sum_{n=0}^{\infty} P(T_i = n)\, z^n = \frac{(1\text{-}P_{ii})\, z}{(1\text{-}P_{ii}\, z)}, \quad |z| < \frac{1}{p_{ii}}$$

$$\bar{T}_i = E(T_i) = \frac{dG(z)}{dz}\Big|_{z=1} = \frac{(1\text{-}P_{ii})^2 + P_{ii}(1\text{-}P_{ii})}{(1\text{-}P_{ii})^2} = \frac{1}{1\text{-}P_{ii}}$$

$$E(T_i^2) = \frac{d^2G(z)}{dz^2}\Big|_{z=1} + \bar{T}_i = \frac{2P_{ii}}{(1\text{-}P_{ii})^2} + \frac{1}{1\text{-}P_{ii}} = \frac{1 + P_{ii}}{(1\text{-}P_{ii})^2}$$

$$\sigma_{T_i}^2 = \frac{P_{ii}}{(1\text{-}P_{ii})^2} \qquad \text{so that } C_{T_i} = \sqrt{P_{ii}} \Rightarrow \text{High } \bar{T}_i \rightarrow \text{High } C_{T_i}$$

DTMC (Geometric pmf) $\leftrightarrow$ CTMC (Exponential density)

CTMC ⇒ Exponential Density

- CTMC: Limiting case of DTMC

Intuitive proof: The process may leave state i even during $(0, \Delta t)$

$$P\{0 \le \tau_i < \Delta t\} = (1-P_{ii}) = -q_{ii}\Delta t$$

$$P\{\Delta t \le \tau_i < 2\Delta t\} = (1-P_{ii})P_{ii} = -q_{ii}\Delta t(1+q_{ii}\Delta t)$$

$$P\{n\Delta t \le \tau_i < (n+1)\Delta t\} = (1-P_{ii})P_{ii}^n = -q_{ii}\Delta t(1+q_{ii}\Delta t)^n$$

$$Let\ \Delta t \to 0 \ \ \&\ \ n \to \infty \ni n\Delta t \to t$$

$$h_{\tau_i}(t) = \lim_{\Delta t \to 0} \frac{P\{n\Delta t \le \tau_i < (n+1)\Delta t\}}{\Delta t} = \lim_{\Delta t \to 0} -q_{ii}(1+q_{ii}\Delta t)^n$$

$$= -q_{ii}(1+q_{ii}\Delta t)^{t/\Delta/}$$

$$= -q_{ii}\, e^{q_{ii}t}$$

$$letting\ \lambda_i = -q_{ii},\quad we\ have\quad h_{\tau_i}(t) = \begin{cases} \lambda_i e^{-\lambda_i t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$CDF: H_{\tau_i}(t) = P(\tau_i \le t) = 1 - e^{-\lambda_i t} \Rightarrow Complementary\ CDF: \overline{H}_{\tau_i}(t) = P(\tau_i > t) = e^{-\lambda_i t}$$

Formal Proof of Exponential Density

- Formal proof:

suppose the process has been in state i for r time units. Want to find

$$P\{\tau_i > r+t \mid \tau_i > r\} = \frac{P(\tau_i > r+t,\ \tau_i > r)}{P(\tau_i > r)} = \frac{P(\tau_i > r+t)}{P(\tau_i > r)}$$

$$\Rightarrow P\{\tau_i > r+t\} = P(\tau_i > r)\ P(\tau_i > r+t \mid \tau_i > r)$$

$$Let\ \bar{H}_{\tau_i}(t) = P(\tau_i > r+t \mid \tau_i > r)\ \forall r\ so\ that\ P(\tau_i > r+t) = p(\tau_i > r)\ \bar{H}_{\tau_i}(t)$$

$$If\ we\ set\ r=0\ and\ noting\ that\ P(\tau_i > 0)=1, we\ have\ \bar{H}_{\tau_i}(t) = P(\tau_i > t)$$

$$So,\ P(\tau_i > r+t) = P(\tau_i > t)\,P(\tau_i > r) = P(\tau_i > t)\,[1-P(\tau_i \le r)]$$

$$P(\tau_i > t) - P(\tau_i > r+t) = P(\tau_i \le r+t) - P(\tau_i \le t) = P(\tau_i > t)\,P(\tau_i \le r)$$

$$H_{\tau_i}(r+t) - H_{\tau_i}(t) = [1-H_{\tau_i}(t)]\,H_{\tau_i}(r) \Rightarrow H_{\tau_i}(r) = \frac{H_{\tau_i}(r+t) - H_{\tau_i}(t)}{[1-H_{\tau_i}(t)]} \Rightarrow h_{\tau_i}(r) = \frac{h_{\tau_i}(r+t)}{1-H_{\tau_i}(t)}$$

$$\Rightarrow h_{\tau_i}(0) = \frac{h_{\tau_i}(t)}{1-H_{\tau_i}(t)} \Rightarrow hazard\ rate = \frac{density}{complementary\ CDF}\ is\ constant$$

$$= -\frac{d}{dt}\ln[1-H_{\tau_i}(t)] \Rightarrow \ln[1-H_{\tau_i}(t)] = -h_{\tau_i}(0)\,t + c \Rightarrow 1-H_{\tau_i}(t) = e^{-h_{\tau_i}(0)t}\cdot e^{c}$$

$$Since\ at\ t=0,\ 1-H_{\tau_i}(0)=1 \Rightarrow c=0 \Rightarrow H_{\tau_i}(t) = 1-e^{-h_{\tau_i}(0)t} \Rightarrow h_{\tau_i}(t) = h_{\tau_i}(0)e^{-h_{\tau_i}(0)t}$$

$$Since\quad h_{\tau_i}(0)\Delta t = [1-p_{ii}] = -q_{ii}\,\Delta t \Rightarrow h_{\tau_i}(0) = -q_{ii} = \lambda_i$$

$$so\quad H_{\tau_i}(t) = 1-e^{q_{ii}t} = 1-e^{-\lambda_i t};\ \mathrm{h}_{\tau_i}(t) = \lambda_i \mathrm{e}^{-\lambda_i t}$$

Moments of Exponential Density

- Moment of exponential density

$$\text{Recall Laplace transforms: } L(s) = \int_0^\infty e^{-st} h_{\tau_i}(t)\, dt$$

$$\left.\begin{aligned} -\frac{dL(s)}{ds}\Big|_{s=0} &= \int_0^\infty t\, h_{\tau_i}(t)\, dt = E(\tau_i) \\ \frac{d^2L(s)}{ds^2}\Big|_{s=0} &= \int_0^\infty t^2\, h_{\tau_i}(t)\, dt = E(\tau_i^2) \end{aligned}\right\rangle \sigma_{\tau_i}^2 = E(\tau_i^2) - [E(\tau_i)]^2$$

$$E(\tau_i^n) = (-1)^n \frac{d^nL(s)}{ds^n}\Big|_{s=0}$$

For exponential density

$$L(s) = \frac{\lambda_i}{s+\lambda_i} \Rightarrow -\frac{dL(s)}{ds} = \frac{\lambda_i}{(s+\lambda_i)^2} \Rightarrow E(\tau_i) = \frac{1}{\lambda_i}$$

$$\frac{d^2L(s)}{ds^2} = \frac{2\lambda}{(s+\lambda)^3} \Rightarrow E(\tau_i^2) = \frac{2}{\lambda_i^2} \Rightarrow \sigma_{\tau_i}^2 = \frac{1}{\lambda_i^2} \Rightarrow C_{\tau_i} = 1$$

$$(-1)^n \frac{d^nL(s)}{ds^n} = \frac{n!\,\lambda}{(s+\lambda)^{n+1}} \Rightarrow E(\tau_i^n) = \frac{n!}{\lambda^n}$$

Uniformization - 1

- Uniformization

  Suppose have a continuous time Markov chain with transition rate matrix $Q = [q_{ij}]$ such that

$$\lambda_i = -q_{ii} = \lambda \quad \forall i$$

$$Define\ P = I + \frac{Q}{\lambda} = \begin{bmatrix} 0 & \frac{q_{12}}{\lambda} & \frac{q_{13}}{\lambda} & \cdots & \frac{q_{1n}}{\lambda} \\ \frac{q_{21}}{\lambda} & 0 & \frac{q_{23}}{\lambda} & \cdots & \frac{q_{2n}}{\lambda} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{q_{n1}}{\lambda} & \cdots & \cdots & \frac{q_{nn-1}}{\lambda} & 0 \end{bmatrix}$$

- # of transitions by time t $\{N(t), t \geq 0\}$ is Poisson with rate λ
- Computationally useful, since can truncate summation at finite k

$$Then,\ P\ is\ a\ transition\ probability\ matrix\ with\ P_{ii} = 0; P_{ij} = \frac{q_{ij}}{\lambda}; i \neq j$$

$$\Phi_{ij}(t) = P\{x(t) = j \mid x(0) = i)\} = (e^{Qt})_{ij}$$

$$But\ e^{Qt} = e^{(\lambda P - \lambda)t} = e^{-\lambda t} \cdot e^{\lambda P t} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda P t)^k}{k!} = \sum_{k=0}^{\infty} P^k \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

Uniformization - 2

- Question: What if $q_{ii} \neq q_{jj}$

Suppose that λ_i are such that $\lambda = \max_k \lambda_k$

$$\text{Form } P^* = I + \frac{Q}{\lambda} = \begin{bmatrix} 1-\frac{\lambda_1}{\lambda} & \frac{q_{12}}{\lambda} & \cdots & \cdots & \frac{q_{1n}}{\lambda} \\ \frac{q_{21}}{\lambda} & 1-\frac{\lambda_2}{\lambda} & \cdots & \cdots & \frac{q_{2n}}{\lambda} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{q_{n1}}{\lambda} & \cdots & \cdots & \frac{q_{nn-1}}{\lambda} & 1-\frac{\lambda_n}{\lambda} \end{bmatrix}$$

Then P is a transition probability matrix with

$$P^*_{ii} = 1-\frac{\lambda_i}{\lambda}; P^*_{ij} = \frac{q_{ij}}{\lambda} = \frac{\lambda_i}{\lambda} P_{ij} \Rightarrow P_{ij} = \frac{q_{ij}}{\lambda_i}$$

$$Note: \text{Real } DTMC: P_{ii} = 0; P_{ij} = \frac{P^*_{ij}}{(1-P^*_{ii})}$$

$$But \; e^{Qt} = e^{(\lambda P^* - \lambda)t} = e^{-\lambda t} \cdot e^{\lambda P^* t}$$

$$= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda P^* t)^k}{k!} = \sum_{k=0}^{\infty} P^{*k} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Real process leaves state i at rate λ_i. But, this is equivalent to saying that transitions occur at rate λ, but only the fraction λ_i/λ of transitions are real ones (and these real transitions occur at rate λ_i) and the remaining $[1-(\lambda_i/\lambda)]$ fraction of transitions are fictitious self-transitions, which leave process in state i.

Summary

- Discrete-time Markov Chains $\Rightarrow$ geometric holding time pmf
- Continuous-time Markov Chains $\Rightarrow$ exponential holding time density
- Poisson process
- Uniformization