Lecture 3

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ECE 336
Stochastic Models for the Analysis of Computer Systems and Communication Networks

Outline of Lecture 3

- Summary of Lecture 2
- Properties of Exponential and Geometric Distributions
- Properties of the Poisson Process
- Steady-state solutions to DTMC and CTMC
- State Transition Rate Diagrams
- Applications to M/M/1,M/M/1/N,M/M/m,M/M/∞,M/M/m/m Queues
- Performance measures for M/M/1, M/M/∞, and M/M/m Queues

Summary of Lecture 2

- DTMC:
  - Specified by

$$P(X_{n+1} = j / X_n = i) = P_{ij}$$

→ *function of n* — non-homogeneous

→ *Independent of n* — homogeneous

  - Unconditional probabilities

$$\underline{p}(n+1) = P^T \underline{p}(n); \ \sum_{j=0}^{N} P_{ij} = 1 \ \forall i = 0,1,\ldots\ldots,N$$

In the steady state, $\underline{p} = P^T \underline{p}$ & $\lim_{n\to\infty}(P^T)^n = [\underline{p}\ \underline{p} \ldots \underline{p}]$; also $\sum_{i=0}^{N} p_i = 1$

The *pmf* of #of stages spent in a given state i: $P(\tau_i = n) = \underbrace{(1-P_{ii})P_{ii}^{n-1}}_{\text{Geometric pmf}}$

- CTMC:
  - Specified by

$$P(X(t+\Delta t) = j / X(t) = i) = q_{ij}(t)\Delta t + o(\Delta t), i \neq j; \ P(X(t+\Delta t) = i / X(t) = i) = 1 + q_{ii}(t)\Delta t + o(\Delta t)$$

$$\sum_{j=0}^{N} q_{ij}(t) = 0 \ \Rightarrow \ q_{ii}(t) < 0 \ \text{since } q_{ij}(t) \geq 0 \Rightarrow Q\underline{e} = \underline{o}$$

$\dot{\underline{p}} = Q^T \underline{p}$, $p_i(t) = \text{Prob}\{X(t) = i\}$; $Q(t)$ time invariant $\Rightarrow$ homogeneous Markov chain

In the staedy state, $\dot{\underline{p}} = \underline{o} = Q^T \underline{p}$

The density of time spent in a given state i: $h_{\tau_i}(t) = \lambda_i \exp(-\lambda_i(t))$, $\lambda_i = -q_{ii}$

Issues Addressed in Lecture 3

- Questions:
  - What are the properties of exponential density & Geometric pmf ?
  - Since inter arrival times of a Poisson distribution are exponentially distributed, what are the properties of Poisson process ?
  - Is there a general way of looking at arrival and departure processes? …. Renewal processes
  - When does steady-state solutions exist for DTMC and CTMC?
  - Are there simple ways of visualizing state transitions?
  - How to analyze simple queues ?

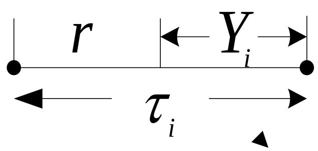
Exponential Density and Geometric pmf -1

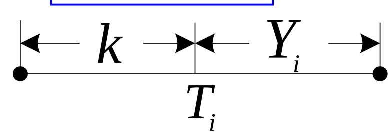
- Properties of Exponential pdf and Geometric pmf

a) Memory-less Property:

Suppose that we have already spent in state *i* for *r* time units (*k* steps). want:

CTMC	DTMC





$$
\begin{aligned}
P(Y_i > t \mid \tau_i > r) &= P(\tau_i > r + t \mid \tau_i > r) \\
&= \frac{P(\tau_i > r + t, \tau_i > r)}{P(\tau_i > r)} \\
&= \frac{P(\tau_i > r + t)}{P(\tau_i > r)} \\
&= e^{-\lambda_i t} \\
&= P(\tau_i > t)
\end{aligned}
$$

Unconditional pdf

$$
\begin{aligned}
&P(Y_i > n \mid T_i > k) \quad n = 1,2,\ldots \\
&= P(T_i > n + k \mid T_i > k) \\
&= \frac{P(T_i > n + k, T_i > k)}{P(T_i > k)} \\
&= \frac{P(T_i > n + k)}{P(T_i > k)} = P_{ii}^n \\
&= P(T_i > n)
\end{aligned}
$$

Unconditional pmf

Exponential Density and Geometric pmf -2

Implication: Suppose τ denotes the time to failure of a component. Result says that if the component has lasted until time r, the distribution of the remaining (residual) time $Y = \tau - r$ is the same as the time to failure of a new component. If the time to failure is exponentially distributed, then a used component is "as good as new".

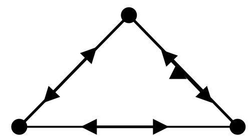
b) Hazard Rate:

$$Z_i(t) = \frac{h_{\tau_i}(t)}{1 - H_{\tau_i}(t)} = \frac{\lambda_i e^{-\lambda_i t}}{e^{-\lambda_i t}} = \lambda_i; \qquad Z_n = \frac{P(T_i = n)}{1 - P(T_i \geq n)} = \frac{P(T_i = n)}{1 - P(T_i > n-1)}$$

$$= 1 - P_{ii}$$

Hazard rate is a constant

Markovian property (e.g., Poisson)



Exponential density

Constant hazard rate

Exponential Density and Geometric pmf - 3

c) Among all the densities having the same mean, exponential density has the largest entropy.

$$H = -E\{\ln h(\tau)\} = -\int_0^\infty h(\tau)\ln h(\tau)\,d\tau$$

Define Lagrangian :

$$L = -\int_0^\infty h(\tau)\ln h(\tau)\,d\tau - \gamma\left[\int_0^\infty h(\tau)\,d\tau - 1\right] - \beta\left[\int_0^\infty \tau h(\tau)\,d\tau - \bar{\tau}\right]$$

$$-\ln h(\tau) - (1+\gamma) - \beta\tau = 0 \Rightarrow h(\tau) = e^{-(1+\gamma+\beta\tau)}$$

From normalization & mean constraints : $\dfrac{e^{-(1+\gamma)}}{\beta} = 1 \Rightarrow e^{-(1+\gamma)} = \beta \Rightarrow h(\tau) = \beta e^{-\beta\tau}$

Mean constraint yields : $\displaystyle\int_0^\infty \beta\tau e^{-\beta\tau}\,d\tau = \bar{\tau} = \frac{1}{\beta} \Rightarrow \beta = \frac{1}{\bar{\tau}}$. so, $\quad h(\tau) = \dfrac{1}{\bar{\tau}} e^{-\tau/\bar{\tau}}$

Can show similar result for geometric pmf : $H = -\displaystyle\sum_{n=0}^{\infty} P(\tau_i = n)\ln P(\tau_i = n)$

Exponential Density and Geometric pmf - 4

d) Suppose have n independent random variables $\tau_1, \tau_2, \ldots, \tau_m$ with parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$, respectively. Then

$$\tau = \min(\tau_1, \tau_2, \ldots, \tau_m) \text{ is exponential with parameter } \sum_{i=1}^{m} \lambda_i$$

Proof:

$$P(\tau \geq t) = P(\tau_1 \geq t) P(\tau_2 \geq t) \ldots P(\tau_m \geq t) = e^{-(\sum_{i=1}^{m} \lambda_i)t}$$

$$\Rightarrow \text{exponential with parameter } \lambda = \sum_{i=1}^{m} \lambda_i$$

Similarly for geometric pmf :

$$T = \min(T_1, T_2, \ldots, T_m) \text{ where } T_i = \text{Geometric with parameter } P_{ii}$$

$$\text{Then, } P(T > n) = P(T_1 > n) P(T_2 > n) \ldots P(T_m > n) = \left(\prod_{i=1}^{m} P_{ii}^{n}\right) = \left(\prod_{i=1}^{m} P_{ii}\right)^{n}$$

$$\therefore T \text{ is geometric with parameter } P = \left(\prod_{i=1}^{m} P_{ii}\right) \Rightarrow P(T = n) = (1 - P) P^{n-1}$$

$$Aside: P(\tau_i < \tau_j) = \int_0^{\infty} P(\tau_i < \tau_j \mid \tau_i = t) h_{\tau_i}(t) dt = \lambda_i \int_0^{\infty} e^{-\lambda_j t} e^{-\lambda_i t} dt$$

$$= \frac{\lambda_i}{\lambda_i + \lambda_j} \Rightarrow \text{relative rate}$$

Exponential Density and Geometric pmf - 5

e) Suppose have *n* independent random variables $\tau_1, \tau_2, \ldots, \tau_m$ with the *same* parameter λ. Then $S_m = \sum_{i=1}^{m} \tau_i$ is $Erlang\text{-}m$ (or $Gamma(m,\lambda)$) $distribution$

Proof:

$$\text{Recall MGF}: L_{\tau_i}(s) = \frac{\lambda}{(s+\lambda)}$$

$$\Rightarrow L_{S_m}(s) = \left(\frac{\lambda}{s+\lambda}\right)^m \Rightarrow h_{S_m}(t) = \frac{\lambda(\lambda t)^{m-1}}{(m-1)!} e^{-\lambda t} \; ; t \geq 0$$

$$\Rightarrow E(S_m) = \frac{m}{\lambda} \; ; E(S_m^2) = \frac{m(m+1)}{\lambda^2}$$

$$\Rightarrow \sigma_{S_m}^2 = \frac{m}{\lambda^2} \Rightarrow C_{S_m} = \frac{1}{\sqrt{m}} \to 0 \; as \; m \to \infty$$

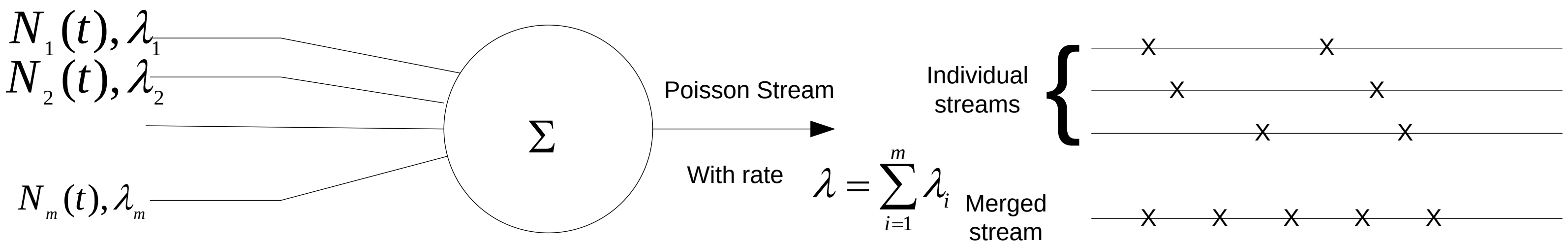
Application:
Pdf of the time until m^{th} arrival.
X(t) Poission $\Leftrightarrow$ *X(t)*= *max{m:* $S_m \leq t$*}* & $S_0 = 0$.

Aside: Suppose $\tau_1, \tau_2, \ldots, \tau_m$ have different parameters, $\lambda_1, \lambda_2, \ldots, \lambda_m ; \lambda_i \neq \lambda_j$

$$\Rightarrow L_{S_m}(s) = \prod_{i=1}^{m} \frac{\lambda_i}{s+\lambda_i} \Rightarrow h_{S_m}(t) = \sum_{i=1}^{m} \lambda_i e^{-\lambda_i t} \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{\lambda_j}{\lambda_j - \lambda_i} \; ; t \geq 0$$

Properties of Poisson Process - 1

a) Merging of Poisson Processes results in a Poisson Process



Proof:

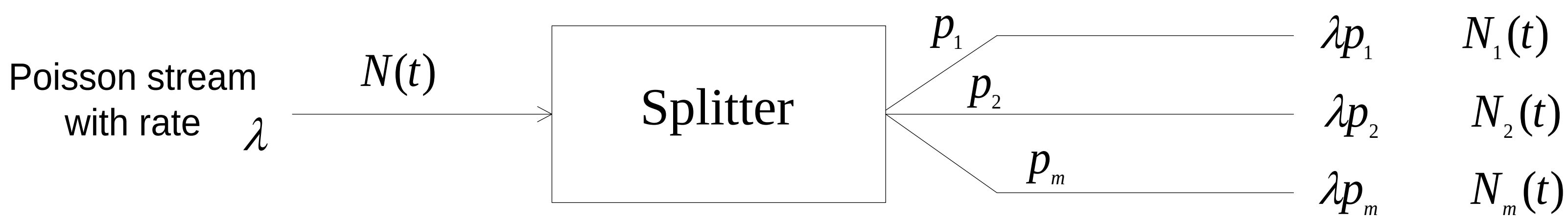
$$N(t) = N_1(t) + \ldots.. + N_m(t)$$

sum of i.i.d. $\Rightarrow$ convolution $\Rightarrow$ product of MGFs

$$G_N(z,t) = e^{(\sum_{i=1}^{m} \lambda_i) t (z-1)} = e^{\lambda t (z-1)} \Rightarrow p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; n = 0,1,2\ldots$$

Properties of Poisson Process - 2

b) Splitting or Decomposition of a Poisson Process



$$N(t) = N_1(t) + N_2(t) + \ldots + N_m(t)$$

$$P(N_1, N_2, \ldots, N_m \mid N) = \frac{N!}{N_1! N_2! \ldots N_m!} p_1^{N_1} p_2^{N_2} \ldots p_m^{N_m}$$

$$P(N_1, N_2, \ldots, N_m; N) = P(N_1, N_2, \ldots, N_m \mid N) P(N)$$

$$= e^{(-\sum \lambda p_i)t} N! \frac{p_1^{N_1} p_2^{N_2} \ldots p_m^{N_m}}{N_1! N_2! \ldots N_m!} \frac{(\lambda t)^{N_1 + N_2 + \ldots + N_m}}{N!}$$

$$= \prod_{i=1}^{m} \frac{e^{-\lambda p_i t} (\lambda p_i t)^{N_i}}{N_i!}$$

$\Rightarrow$ Individual streams are independent and poisson with rates, λp_i

Properties of Poisson Process - 3

c) Conditional Distribution of the Arrival Times

Suppose one event of a Poisson process has taken place by time, t. Given this information, the distribution of the time at which the event occurred is *uniform* in $[0,t]$.

$$P(\tau_1 < s \mid N(t)=1) = \frac{P(\tau_1 < s, N(t)=1)}{P(N(t)=1)}$$

$$= \frac{P\{1 \text{ event in } [0,s), 0 \text{ events in } [s,t)\}}{P(N(t)=1)}$$

$$= \frac{P\{1 \text{ event in } [0,s)\} P\{0 \text{ events in } [s,t)\}}{P(N(t)=1)}$$

$$= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t} \Rightarrow \text{uniform}$$

Aside: Given that $N(t)=n$, the n arrival times $S_1, S_2, \ldots, S_n$ are i.i.d. and uniformly distributed in the interval $[0,t]$

Inter-arrival times: $\tau_i = S_i - S_{i-1}; S_0 = 0$

$$P(s_1, s_2, \ldots, s_n \mid n) = \frac{h(s_1, s_2, \ldots, s_n, n)}{P(N(t)=n)} = \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \ldots e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t-s_n)}}{e^{-\lambda t}(\lambda t)^n / n!}$$

$$= \frac{n!}{t^n}; \; 0 < s_1 < \ldots < s_n < t$$

Properties of Poisson Process - 4

d) Poisson pmf is a Special Case of Binomial pmf

$$p(n) = \binom{N}{n} p^n (1-p)^{N-n}$$

$$\left.\begin{aligned} &\text{Let} \quad p = \lambda \Delta t \\ &\qquad N = \frac{t}{\Delta t} \\ &\lim_{\Delta t \to 0} Np = \lambda t \end{aligned}\right\} \Rightarrow p(n) = \frac{N!}{N-n!n!}(\lambda \Delta t)^n (1-\lambda \Delta t)^{N-n} = \frac{N!}{N-n!n!}\frac{(\lambda t)^n}{N^n}(1-\frac{\lambda t}{N})^{N-n}$$

$$= \frac{N(N-1)\ldots(N-n+1)}{n!N^n}(\lambda t)^n (1-\frac{\lambda t}{N})^{\frac{N}{\lambda t}\cdot\frac{\lambda t(N-n)}{N}}$$

$$= \frac{N(N-1)\ldots(N-n+1)}{n!N^n}(\lambda t)^n e^{-\lambda t\frac{N-n}{N}}$$

$$\text{As} \quad N \to \infty \qquad p(n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; \;\; n = 0,1,2,\ldots$$

Digression: Renewal Processes

- Suppose $\{\tau_i : i = 0,1,2,,,\}$ are i.i.d. nonnegative random variables (*not necessarily exponential*). Let $S_n = \sum_{i=1}^{n} \tau_i$

$$S_{n+1} = S_n + \tau_{n+1} \Rightarrow h_{S_{n+1}}(t) = h_{S_n}(t) * h_\tau(t) = \int_0^t h_{S_n}(t-\sigma) h_\tau(\sigma) d\sigma$$

$X(t) \sim$ counting process

$$P(X(t) = n) = P\{S_n \le t < S_{n+1}\} = P(S_n \le t) - P(S_{n+1} \le t) = H_{S_n}(t) - H_{S_{n+1}}(t)$$

$$\Rightarrow P(X(t) \ge n) = H_{S_n}(t) \Rightarrow M(t) = E[X(t)] = \sum_{n=1}^{\infty} H_{S_n}(t) = H_\tau(t) + \sum_{n=1}^{\infty} H_{S_{n+1}}(t)$$

$$\Rightarrow M(t) = H_\tau(t) + \sum_{n=1}^{\infty} \int_0^t H_{S_n}(t-\sigma) h_\tau(\sigma) d\sigma = H_\tau(t) + \int_0^t \sum_{n=1}^{\infty} H_{S_n}(t-\sigma) h_\tau(\sigma) d\sigma$$

$$\Rightarrow M(t) = H_\tau(t) + \int_0^t M(t-\sigma) h_\tau(\sigma) d\sigma$$ **Fundamental Renewal Equation**

$$Let\ \ Rate\ of\ \ renewals,\ m(t) = \frac{dM(t)}{dt} = \sum_{n=1}^{\infty} h_{S_n}(t) \Rightarrow m(t) = h_\tau(t) + \int_0^t m(t-\sigma) h_\tau(\sigma) d\sigma$$ **Renewal Equation**

$$\text{In Laplace domain}: L_m(s) = \frac{L_\tau(s)}{1 - L_\tau(s)} \Rightarrow \lim_{t\to\infty} m(t) = \lim_{s\to 0} s L_m(s)$$

$$\lim_{t\to\infty} m(t) = \frac{L_\tau(s) + s\frac{dL_\tau(s)}{ds}}{-\frac{dL_\tau(s)}{ds}}\Bigg|_{s=0} = \frac{1}{E[\tau]} \Rightarrow \text{long term rate of renewals} = \frac{1}{\text{inter - renewal time}}$$

Poisson Process is a Renewal Process

- *Suppose* $\{\tau_i : i = 0,1,2,,,\}$ *are i.i.d. exponential random variables*

Let $S_n = \sum_{i=1}^{n} \tau_i$

$$Exponential \Rightarrow L_\tau(s) = \frac{\lambda}{(s+\lambda)}$$

$$\Rightarrow L_m(s) = \frac{\lambda}{s} \Rightarrow m(t) = \lambda$$

$$\Rightarrow M(t) = \lambda t = \text{Expected number of renewals in } [0,\text{t}] = \textit{mean of Poisson}$$

$$Also,\ L_{S_n}(s) = [L_\tau(s)]^n \Rightarrow h_{S_n}(t) = Erlang(n,\lambda) = \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}$$

$$\Rightarrow H_{S_n}(t) = \int_0^t h_{S_n}(\sigma) d\sigma = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$\Rightarrow P(X(t) = n) = H_{S_n}(t) - H_{S_{n+1}}(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}; n = 0,1,2,..$$

Steady-state Probabilities -1

- When does steady-state solution to $\underline{p} = P^T \underline{p}$ & $\underline{0} = Q^T \underline{p}$ exist?

  We need *n*-step transition probabilities. Let

$$\Phi_{ij}(n) = P\{X_{m+n} = j / X_m = i\} = P\{X_n = j / X_0 = i\} \quad \text{time-homogeneous MC}$$

$$\Phi_{ij}(n) = \sum_{k=0}^{N} P\{X_n = j / X_{n-1} = k\} . P\{X_{n-1} = k / X_0 = i\}$$

$$= \sum_{k=0}^{N} \Phi_{ik}(n-1) P_{kj}$$

$$\Phi(n) = \Phi(n-1) P \quad \Rightarrow \quad \Phi(n) = P^n \text{ Similar to state-transition matrix in control theory}$$

- Two states i and j communicate if $\exists\, n, n' \ni \Phi_{ij}(n) = (P^n)_{ij} > 0 \text{ and } \Phi_{ji}(n') = (P^{n'})_{ji} > 0$
  - state *i* communicates with itself
  - if *i* and *j* communicate, then *j* and *i* communicate
  - if *i* and *j* communicate, and *j* and *k* communicate, then *i* and *k* communicate (transitive)
  - communicating states define a *class* $\Rightarrow$ comm. divides states into *classes*
- If <u>all</u> states communicate (i.e., one class) $\Rightarrow$ Markov chain is *irreducible*

First Passage Probabilities

- First Passage Probabilities and Recurrence Times

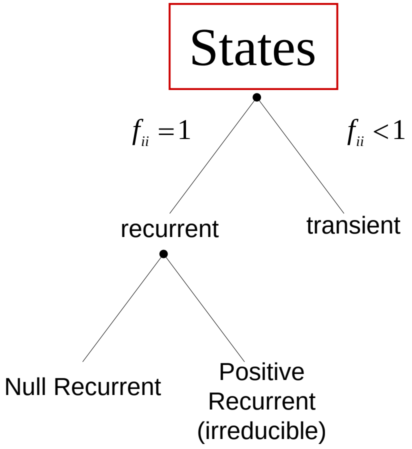
$f_{ij}(n) = P\{first \text{ return to state } j \text{ occurs after n steps of leaving state } i\}$

$= P\{first\ time\ X(n) = j \mid X(0) = i\}$

$\Rightarrow$ State j can be reached for the first time at the k^{th} step with probability $f_{ij}(k)$ and again in the remaining $n-r$ steps with probability $\Phi_{jj}(n-k) \Rightarrow \Phi_{ij}(n) = \sum_{k=1}^{n} f_{ij}(k)\Phi_{jj}(n-k); n \geq 1$

$Initial\ conditions$: $f_{ij}(0) = 0;\ f_{ij}(1) = P_{ij}; \Phi_{jj}(0) = 1, \Phi_{ij}(0) = 0.$

$\Rightarrow$ For $i = j, \Phi_{ii}(z) = 1 + F_{ii}(z)\Phi_{ii}(z) \Rightarrow \Phi_{ii}(z) = \dfrac{1}{1 - F_{ii}(z)}$



$f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n) = \text{Prob}\{\text{ever returning to state } i\}$

$f_{ii} = 1 \Rightarrow\ persistent\ (recurrent);\ f_{ii} < 1 \Rightarrow transient$

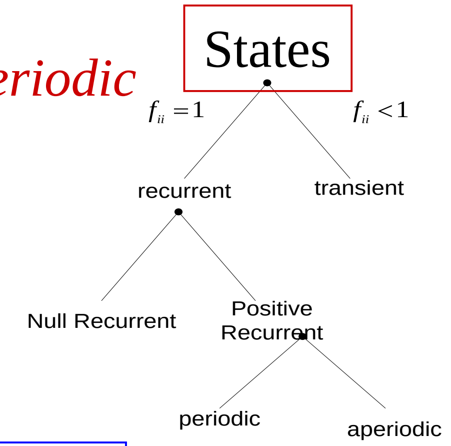
$Mean\ Recurrence\ Time:$

$M_i = \sum_{n=1}^{\infty} n f_{ii}(n)$ $\qquad M_i = \infty$ recurrent $null$

$M_i < \infty$ recurrent nonnull (Positive recurrent)

Steady-state Probabilities - 2

- Further classification of Markov-chain states
  - If the return to state *j* occurs only at $\gamma, 2\gamma, 3\gamma, \ldots$ ($\gamma > 1$), $\{\Phi_{ii}(n) = 0 \text{ unless } n = k\gamma\}$, then state *j* is *periodic*
  - If $\gamma = 1$ for state *i*, $\{\Phi_{ii}(n) > 0\ \forall n\}$, then it is said to be *aperiodic*



If one state in a class is *aperiodic*, all states in that class are
If one state in a class is *transient*, all states in that class are
If one state in a class is *recurrent*, all states in that class are
If one state in a class is *periodic*, all states in that class are

Irreducible $\Rightarrow$ recurrent non-null (positive recurrent) and aperiodic

- Irreducible aperiodic (i.e., ergodic) Markov chains have:

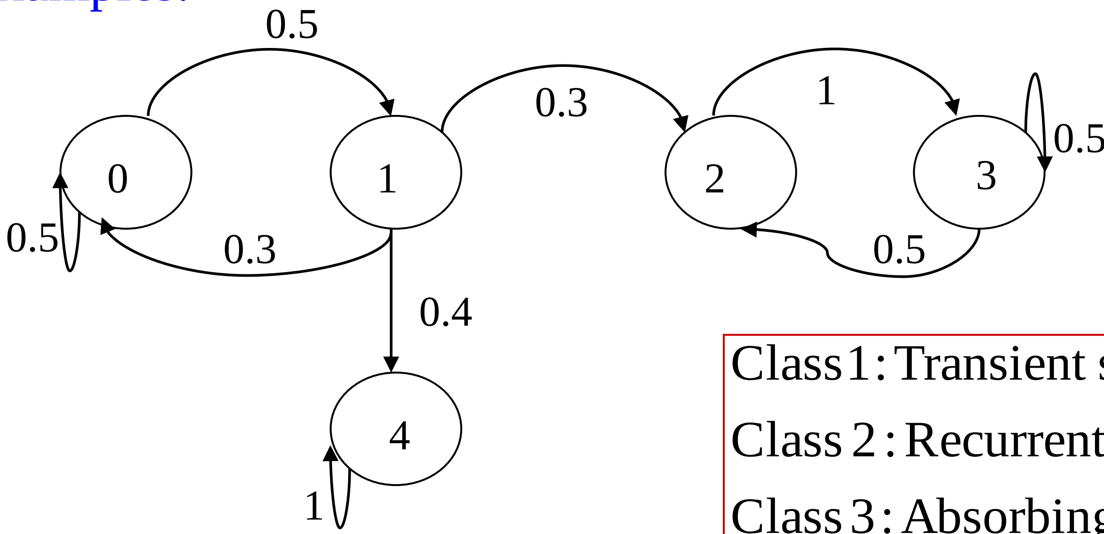
$$1)\ p_i = 0 \quad \forall i \geq 0 \Rightarrow \text{no steady-state distribution (e.g., arrival rate > service capacity)}$$

$$2)\ p_i > 0 \quad \forall i \quad \Rightarrow \{p_i / i \geq 0\} = p_i = \frac{1}{M_i} \text{ is the unique stationary distribution}$$

- Absorbing state: $P_{ii} = 1 \Rightarrow M_i = 1 \Rightarrow p_i = 1$

Examples of State Classification - 1

- Examples:



Class 1 : Transient states (0,1)

Class 2 : Recurrent states (2,3)

Class 3 : Absorbing state (4)

Steady state probabilities can be one of these two :

Two unity eigen values since two recurrent classes

- $[0\ 0\ 0\ 0\ 1] \Rightarrow$ chain is trapped in state 4
- $[0\ 0\ 0.2522\ 0.5044\ 0.2434]$

  $\Rightarrow$ finite probability of getting trapped in class of states (2,3)

Examples of State Classification - 2

- Examples:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \Phi(n) = P^n = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & n \text{ even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & n \text{ odd} \end{cases}$$

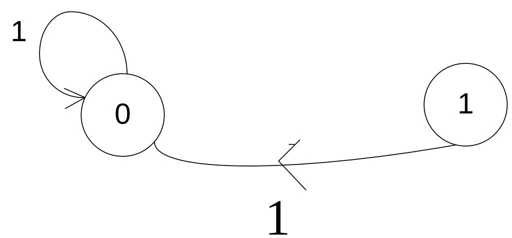
starting in state 0, we return to state 0 in even # of steps

$\Rightarrow \gamma = 2$ *periodic*

$$P = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix} \Rightarrow \{\Phi_{ii}(n) = (P^n)_{ii} > 0 \forall n\}$$

aperiodic Markov chain

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$



State 0: absorbing
State 1: transient

$$steady\ state\ probability : [1 \;\; 0]$$

We will consider *irreducible*, *aperiodic Markov chains*

State Transition Rate Diagrams - 1

$$\dot{p}_i = \sum_{j=0}^{N} q_{ji}(t) p_j(t) = \sum_{\substack{j \neq i \\ j=0}}^{N} q_{ji} p_j(t) + q_{ii}(t) p_i(t)$$

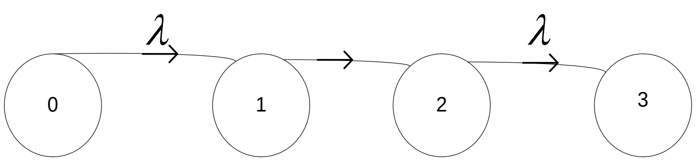
$$= \underbrace{\sum_{\substack{j \neq i \\ j=0}}^{N} q_{ji} p_j(t)}_{\text{FLOW INTO NODE } i} - \underbrace{\sum_{\substack{j \neq i \\ j=0}}^{N} q_{ij} p_i(t)}_{\text{FLOW OUT OF NODE } i} \Rightarrow \text{No need to know } q_{ii}$$

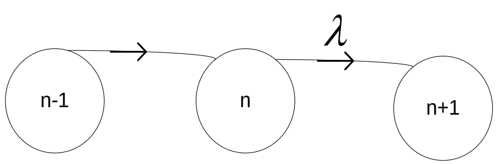
- Examples

- Poisson Process

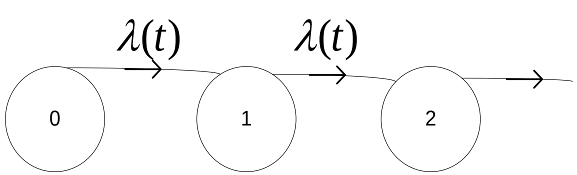
$$\dot{p}_n = \lambda p_{n-1}(t) - \lambda p_n(t)$$

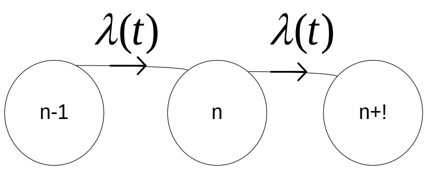
$$\Rightarrow p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$





- Non-homogeneous Poisson process





$$\dot{p}_n = \lambda(t) p_{n-1}(t) - \lambda(t) p_n(t)$$

$$\Rightarrow p_n(t) = \frac{\left[\int_0^t \lambda(\tau) d\tau\right]^n}{n!} e^{-\int_0^t \lambda(\tau) d\tau}$$

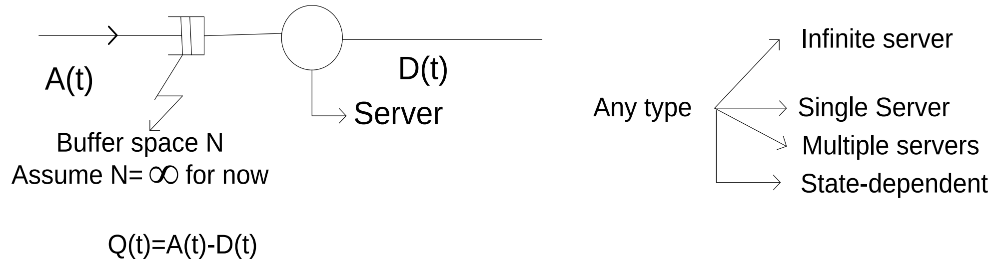
State Transition Rate Diagrams - 2

- State-dependent birth process

$$\dot{p}_n(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t)$$

$$\Rightarrow \quad p_n(s) = \frac{\lambda_{n-1} p_{n-1}(s)}{s+\lambda_n} \Rightarrow p_n(t) = e^{-\lambda_n t}\left[\lambda_{n-1}\int_0^t p_{n-1}(\tau)e^{\lambda_n \tau}d\tau + p_n(0)\right]$$

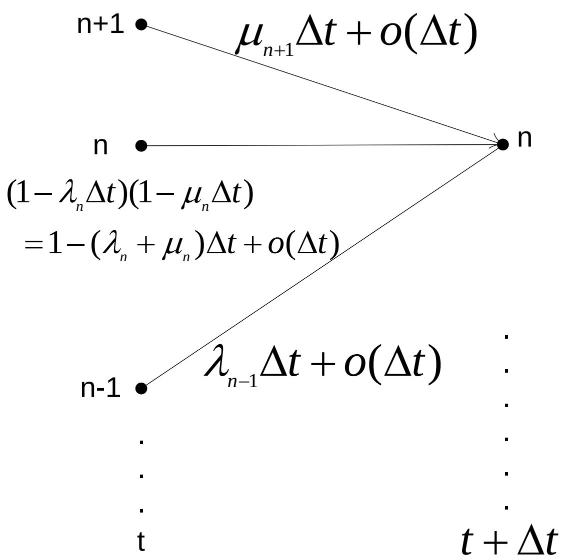
- Birth-death processes: Forms the basis of *all* Markovian Queuing systems



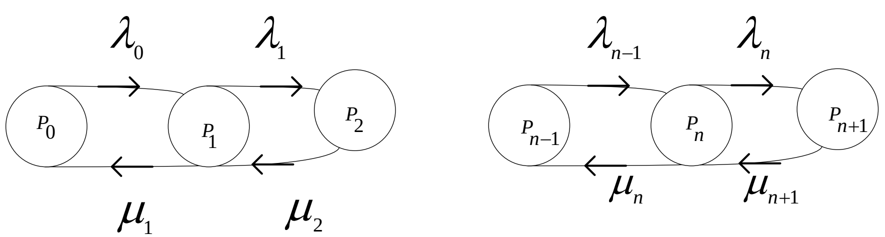
State Transition Rate Diagrams - 3

$A(t) = \text{total births upto time t}$
$D(t) = \text{total deaths upto time t}$
$\Rightarrow Q(t) = \text{system length at time t}$

We assume that in an interval of time Δt only one arrival or departure can occur



We assume S=1 (service demand=1)



$$\dot{p}_n(t) = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t);\ n \geq 1$$
$$\dot{p}_0(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

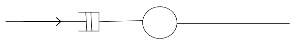
Application to Simple Queues - 1

- B-D processes model a wide variety of Markovian (exponential) queues

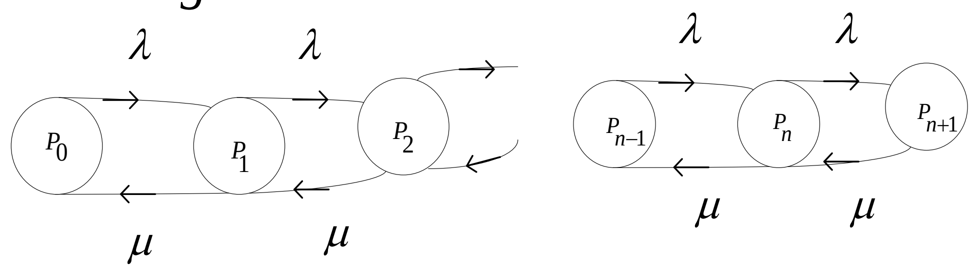
1) M/M/1 Queue   *M ~ Poisson arrivals or exponential inter-arrival pdf*

M ~ exponential service demands

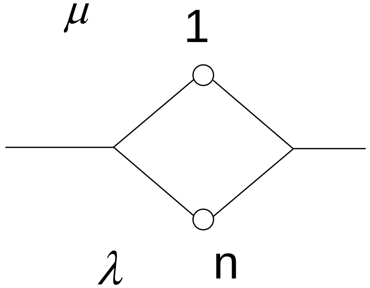
1 ~ single server



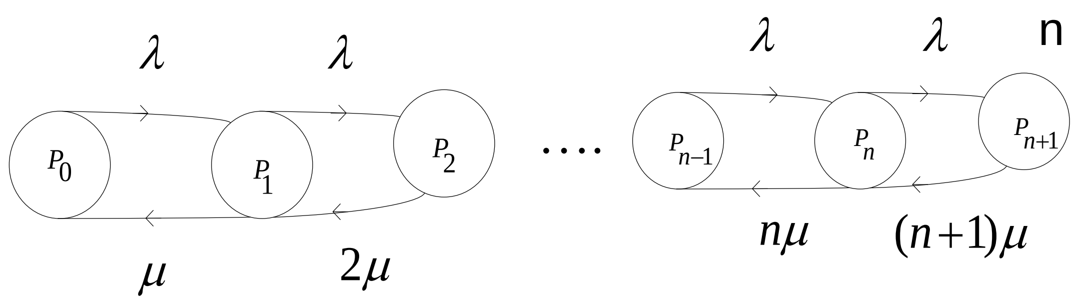
$$\left.\begin{array}{l}\lambda_n = \lambda \\ \mu_n = \mu\end{array}\right\} \Rightarrow$$



2) M/M/∞ queue   Infinite server case ⇒ No waiting

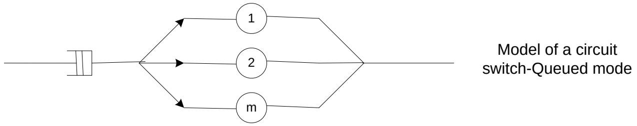


$$\left.\begin{array}{l}\lambda_n = \lambda \\ \mu_n = n\mu\end{array}\right\} \Rightarrow$$



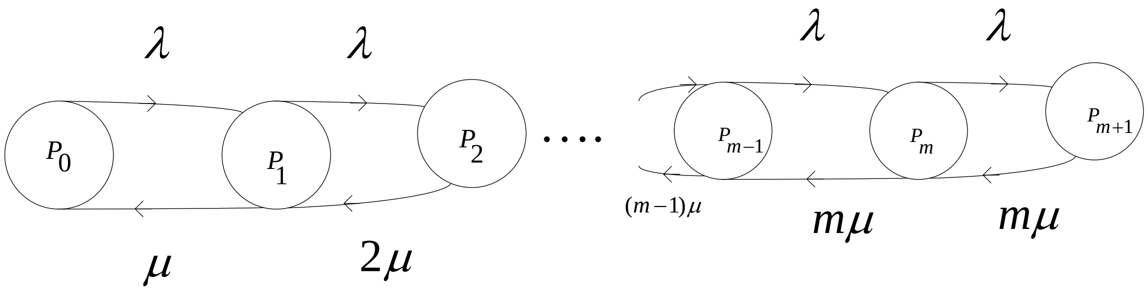
Application to Simple Queues - 2

3) M/M/m queue $\Rightarrow$ m server case

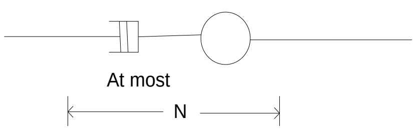


$$\lambda_n = \lambda$$

$$\mu_n = \begin{cases} n\mu & n \le m \\ m\mu & n > m \end{cases} \Rightarrow$$

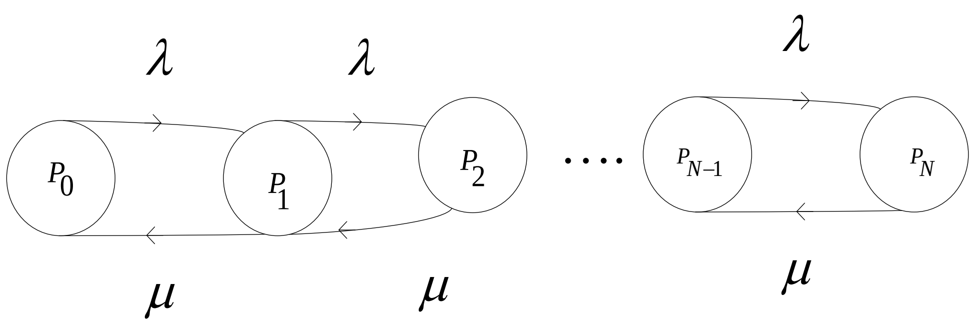


4) M/M/1/N queue $\Rightarrow$ Model of a single machine with a finite buffer



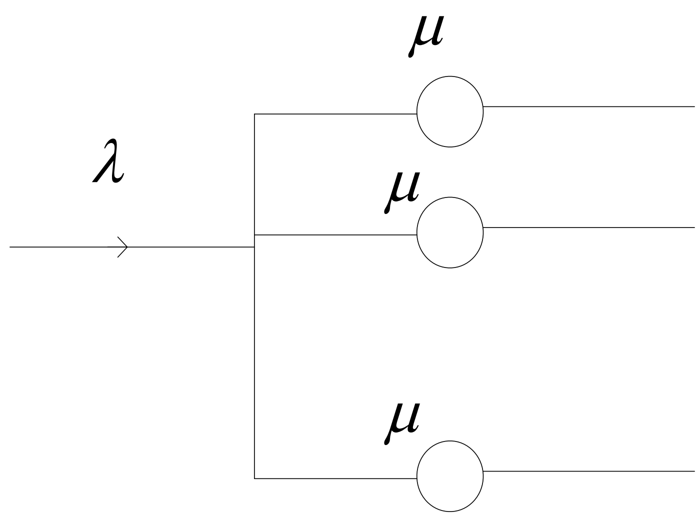
$$\lambda_n = \begin{cases} \lambda & n \le N-1 \\ 0 & n \ge N \end{cases}$$

$$\mu_n = \begin{cases} \mu & n \le N \\ 0 & n > N \end{cases}$$



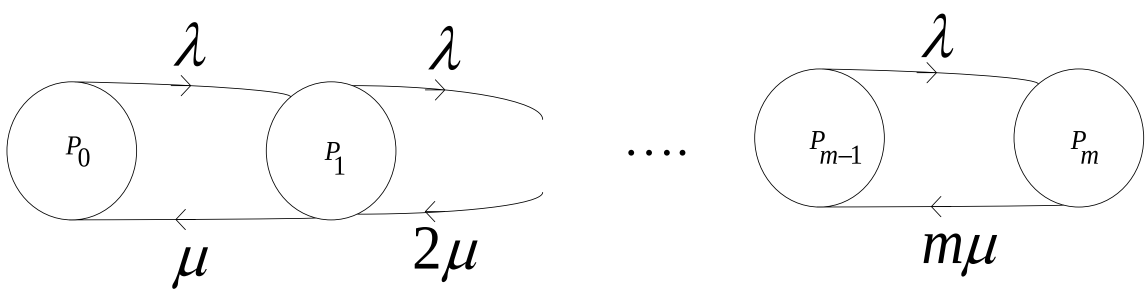
Application to Simple Queues - 3

5) M/M/m/m model⇒Blocked call loss system or Blocked calls cleared (BCC)



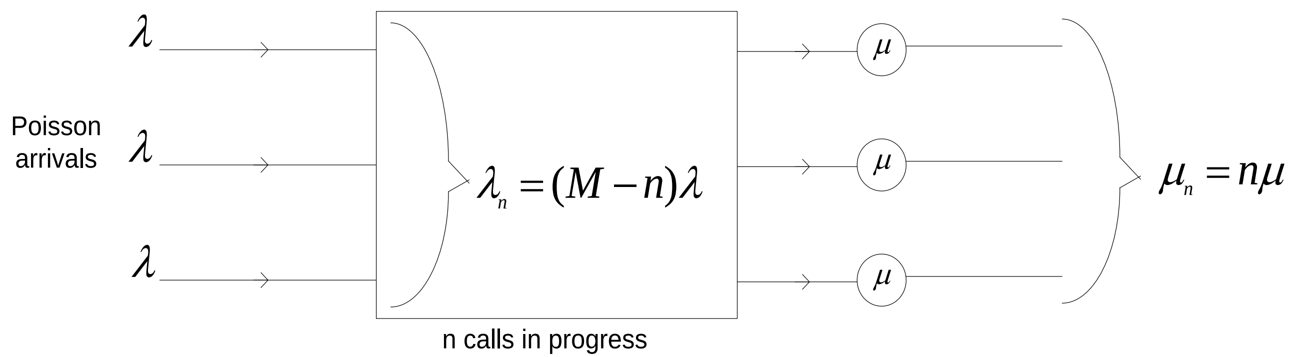
$$\lambda_n = \lambda \quad 0 \le n \le m-1$$

$$\mu_n = \begin{cases} n\mu & n \le m \\ 0 & n > m \end{cases}$$



Application to Simple Queues - 4

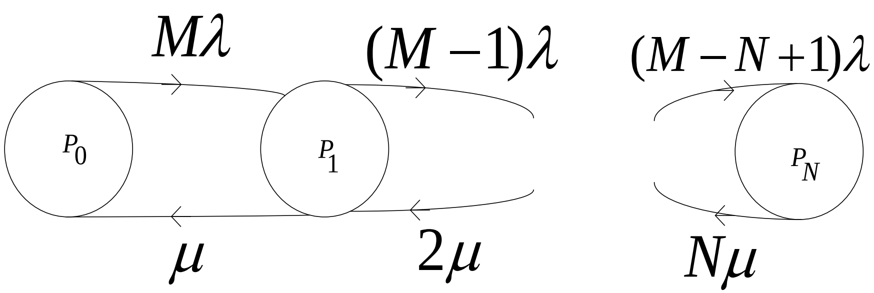
6) Engset model *(M>N) M/M/N/N/M queue*



$$\lambda_n = (M-n)\lambda; \quad 0 \le n \le N$$

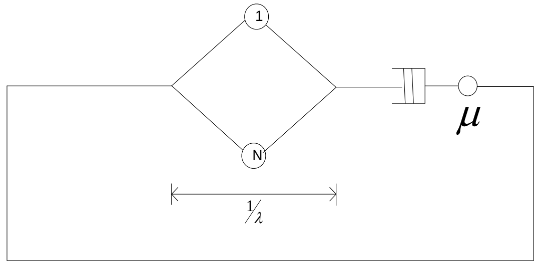
$$N \le M$$

$$\mu_n = n\mu; \quad 1 \le n \le N$$



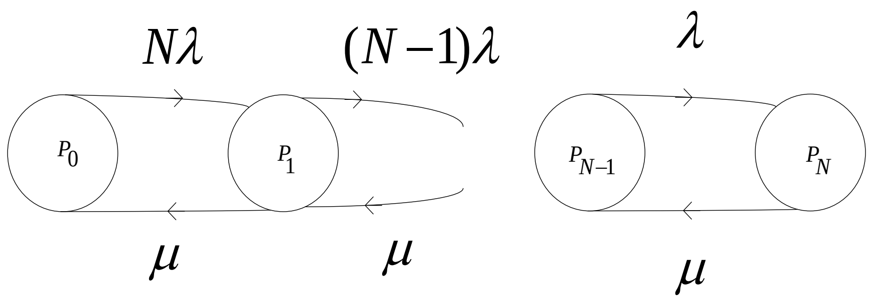
Application to Simple Queues - 5

7) Machine Repairman model….forms the basis of closed networks
M/M/1/N/N queue

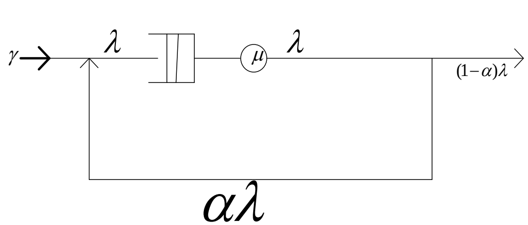


$$\lambda_n = \begin{cases} (N-n)\lambda & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_n = \mu$$



8) M/M/1 queue with feedback …. A simple open network



$$\Rightarrow \lambda_n = \frac{\gamma}{1-\alpha}$$

$$\mu_n = \mu$$

Solution of Birth-Death Model - 1

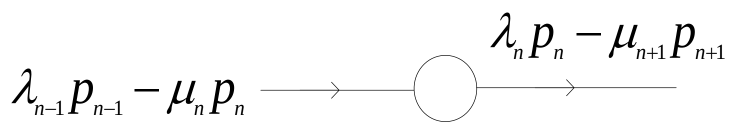
- Coming back to birth-death model

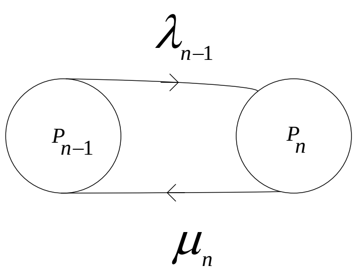
$$p_n(t+\Delta t) = \lambda_{n-1} p_{n-1}(t)\Delta t + [1-(\lambda_n+\mu_n)\Delta t] p_n(t) + \mu_{n+1} p_{n+1}(t)\Delta t$$

$$\left.\begin{aligned} \frac{dp_n(t)}{dt} &= \lambda_{n-1} p_{n-1}(t) - (\lambda_n+\mu_n) p_n(t) + \mu_{n+1} p_{n+1}(t); n \ge 1 \\ \frac{dp_0(t)}{dt} &= -\lambda_0 p_0(t) + \mu_1 p_1(t) \end{aligned}\right\} \text{known as } \textbf{global balance equations}$$

Solution via numerical methods. If $\lambda_n = \lambda, \mu_n = \mu$ (i.e., M/M/1 queue), solution involves Bessel functions. We will be content with steady-state solution only.

$$p_n = \lim_{t\to\infty} p_n(t)$$





$$\Rightarrow \lambda_{n-1} p_{n-1} = \mu_n p_n$$

Local balance
n detailed Balance Equations

Solution of Birth-Death Model - 2

$$\Rightarrow p_n = \frac{\lambda_{n-1}}{\mu_n} p_{n-1} = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} p_0 = \left[\prod_{i=0}^{n-1} \left(\frac{\lambda_i}{\mu_{i+1}} \right) \right] p_0$$

$$\text{Since } \sum_{n=0}^{\infty} p_n = 1 \Rightarrow p_0 = [1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}]^{-1}$$

- Once the distribution is known, we can obtain a variety of performance measures

1) Throughput: $X = \sum_{n=1}^{\infty} \mu_n p_n$

2) System (or Queue) length $Q = \sum_{n=1}^{\infty} n p_n$

3) Utilization $U = \dfrac{\sum_{n=1}^{\infty} \mu_n p_n}{\max_n \mu_n}$

4) Average Response time: $R = \dfrac{Q}{X}$ Little's law

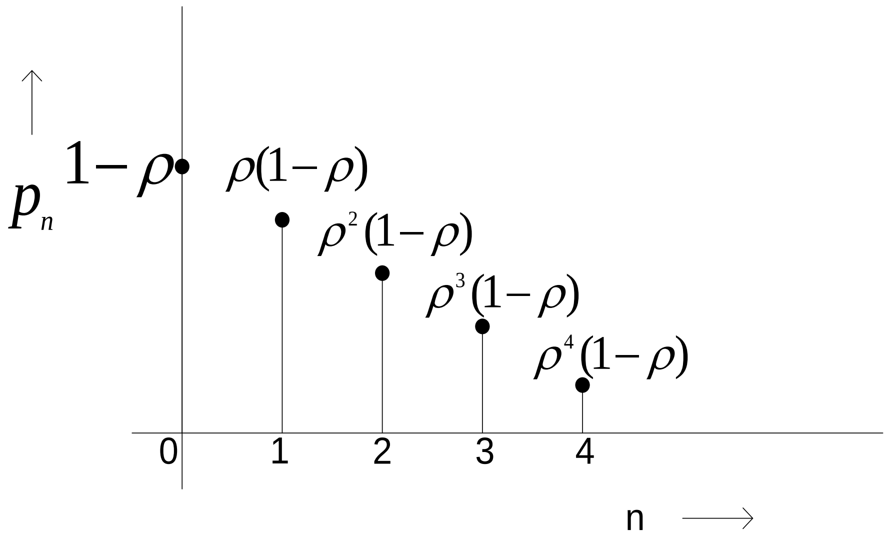
Application to M|M|1 Queue - 1

1. *M/M/1 Queue*

$$\lambda_n = \lambda, \mu_n = \mu \Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n p_0; \frac{\lambda}{\mu} = \rho = \text{traffic intensity}$$

$$p_0 = \left(1 + \sum_{n=1}^{\infty} \rho^n\right)^{-1} = 1 - \rho = 1 - \frac{\lambda}{\mu}; \textit{ Need } \lambda < \mu \text{ or } \rho < 1$$

$$\therefore p_n = (1-\rho)\rho^n \quad n = 0,1,2,\ldots \quad \text{Modified Geometric pmf... note } n \text{ versus } (n-1)$$



Application to M|M|1 Queue - 2

$$G(z) = \sum_{n=0}^{\infty} p_n z^n = \frac{(1-\rho)}{1-\rho z}$$

$$E(n) = Q = \left.\frac{dG(z)}{dz}\right|_{z=1} = \left.\frac{(1-\rho)\rho}{(1-\rho z)^2}\right|_{Z=1} = \frac{\rho}{1-\rho}$$

$$E(n^2) - Q = \left.\frac{d^2G(z)}{dz^2}\right|_{z=1} = \frac{2(1-\rho)\rho^2}{(1-\rho)^3} = \frac{2\rho^2}{(1-\rho)^2}$$

$$\Rightarrow \sigma_n^2 = \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{1-\rho} - \left(\frac{\rho}{1-\rho}\right)^2 = \frac{\rho}{(1-\rho)^2} \Rightarrow \sigma_n = \frac{\sqrt{\rho}}{1-\rho}$$

$$C_n = \frac{\sigma_n}{Q} = \frac{1}{\sqrt{\rho}} \geq 1$$

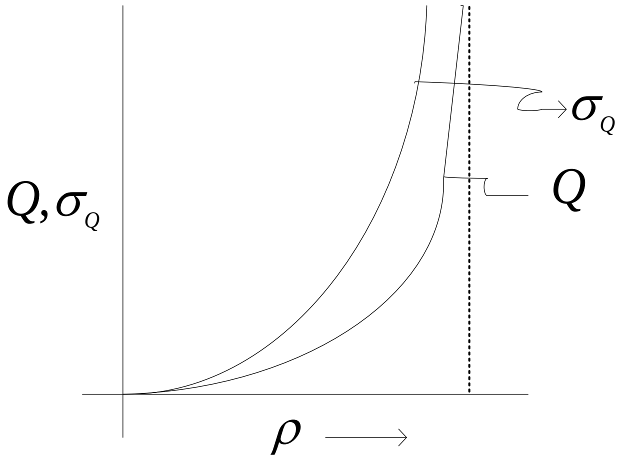
- Throughput $X = \sum_{n=1}^{\infty} \mu_n p_n = \mu(\sum_{n=1}^{\infty} \rho^n)(1-\rho) = \rho\mu = \lambda$
- Average response time $R = \frac{Q}{X} = \frac{1}{\mu(1-\rho)}$

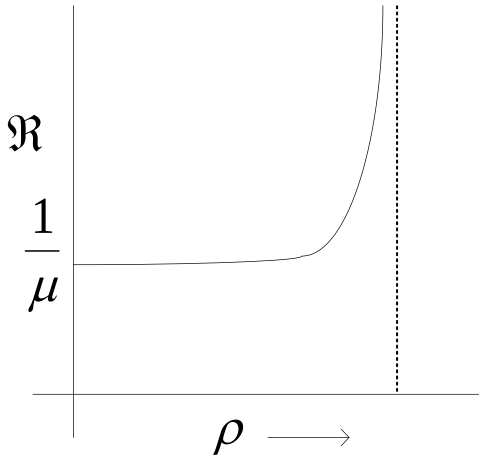
Higher moments of response time requires analysis… later

Application to M|M|1 Queue - 3

- Utilization $U = \dfrac{\lambda}{\mu} = \rho$
- Average waiting time $W = R - \dfrac{1}{\mu} = \dfrac{1}{\mu}\dfrac{\rho}{1-\rho}$
- Average waiting queue length

$$Q_W = XW = \frac{\rho^2}{1-\rho} \left(check : Q_W = Q - \rho = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho} \right)$$





Application to M|M|∞ Queue

2. M/M/∞ Queue $\lambda_n = \lambda, \mu_n = n\mu$

$$p_n = \frac{\lambda}{n\mu} p_{n-1} \Rightarrow p_n = \frac{\rho^n}{n!} p_0; \;\; \rho = \frac{\lambda}{\mu} \Rightarrow p_0 = \left(1 + \sum_{n=1}^{\infty} \frac{\rho^n}{n!}\right)^{-1} = e^{-\rho}$$

$$\text{The result}: \;\; p_n = \frac{\rho^n}{n!} e^{-\rho}; n = 0,1,2,\ldots$$

POISSON DISTRIBUTION WITH MEAN ρ

$$\left.\begin{aligned} Q &= \rho \\ \sigma_n &= \sqrt{\rho} \end{aligned}\right\} \Rightarrow C_n = \frac{1}{\sqrt{\rho}}$$

$$\text{Throughput} = X = \sum_{n=1}^{\infty} \mu_n p_n = \sum_{n=1}^{\infty} \frac{n\mu.\rho^n}{n!} e^{-\rho} = \mu\rho \left(\sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!} \right) e^{-\rho} = \lambda \text{ as it should}$$

$$R = \frac{1}{\mu}$$

$$W = R - \frac{1}{\mu} = 0$$

$$Q_W = 0$$

$$U = \lim_{n\to\infty} \frac{\lambda}{n\mu} = 0$$

Results valid for $M|G|\infty$ queue

Application to M|M|m Queue -1

1. M/M/m Queue

$$p_n = \frac{\lambda}{n\mu} p_{n-1} \quad \text{for} \quad 1 \le n \le m \Rightarrow p_n = \prod_{i=0}^{n-1} \left(\frac{\lambda}{(i+1)\mu} \right) p_0 = \left(\frac{\lambda}{\mu} \right)^n \frac{1}{n!} p_0 \quad 1 \le n \le m$$

$$\text{For } n > m: p_n = \frac{\lambda}{m\mu} . p_{n-1} \Rightarrow p_n = \left(\frac{\lambda}{m\mu} \right)^{n-m} p_m ; n > m$$

$$\underline{\text{Result}}: p_n = \begin{cases} \dfrac{\left(\frac{\lambda}{\mu}\right)^n}{n!} p_0 & 0 \le n \le m \\ \dfrac{\left(\frac{\lambda}{\mu}\right)^m}{m!} \left(\dfrac{\lambda}{m\mu} \right)^{n-m} p_0 & n > m \end{cases} \qquad \text{From} \quad \sum_{n=0}^{\infty} p_n = 1, \quad \left[\sum_{n=0}^{m-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \sum_{n=m}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!} \left(\frac{\lambda}{m\mu} \right)^{n-m} \right] p_0 = 1$$

$$\Rightarrow p_0 = \left[\sum_{n=0}^{m-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!} \frac{1}{\left(1 - \frac{\lambda}{m\mu}\right)} \right]^{-1}$$

Application to M|M|m Queue - 2

- Probability of Queuing, P_Q

$$P_Q = \sum_{n=m}^{\infty} p_n = \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!} p_0 \sum_{n=m}^{\infty} \left(\frac{\lambda}{m\mu}\right)^{n-m} = \frac{p_m}{1-\frac{\lambda}{m\mu}} = \frac{p_m}{1-\rho}; \rho = \frac{\lambda}{m\mu}$$

(Erlang's delay formula or Erlang's C - formula or Erlang's second formula)

- Average Number of Customers Waiting (not in service)

$$Q_W = \sum_{n=0}^{\infty} n p_{n+m} = \sum_{n=0}^{\infty} n \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!} \rho^n p_0 = p_m \sum_{n=0}^{\infty} n\rho^n = p_m \rho . \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{1-\rho}\right) = \frac{\mathrm{p_m}\rho}{(1-\rho)^2} = \frac{\rho P_Q}{(1-\rho)}$$

$$\Rightarrow \frac{Q_W}{P_Q} = \frac{\rho}{1-\rho} \quad ; \rho = \frac{\lambda}{m\mu}$$

- Throughput

$$X = \sum_{n=1}^{\infty} p_n \mu_n = m\mu \sum_{n=m}^{\infty} p_n + \sum_{n=1}^{m-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} . n\mu . p_0 = \lambda \sum_{n=m}^{\infty} p_{n-1} + \lambda \sum_{n=0}^{m-2} p_n = \lambda \text{ as } \underline{\text{it should}}$$

Application to M|M|m Queue - 3

- Average Waiting Time $W = \frac{Q_W}{\lambda} = \frac{1}{\mu}\frac{P_Q}{1-\rho}$
- Average Response Time $R = W + \frac{1}{\mu} = \frac{1}{\mu}[1 + \frac{P_Q}{1-\rho}]$
- Average Queue Length $Q = R.\lambda = \frac{\lambda}{\mu}[1 + \frac{P_Q}{1-\rho}]$
- Average number of busy servers $Q - Q_w = \frac{\lambda}{\mu}$

Note $\frac{\lambda}{\mu}$ can be >1 but $\frac{\lambda}{m\mu}$ must be <1

- An alternative expression for system length … a trick that will be useful for state-dependent nodes of networks

$$p_n = \frac{\lambda}{\mu(n)} p_{n-1}$$

Multi – server

$$\mu(n) = \begin{cases} n\mu & n \le m \\ m\mu & n > m \end{cases}$$

State - dependent node

$$\{\mu(1)....\mu(m)\}$$

Application to M|M|m Queue - 4

- Queue Length

$$Q = \sum_{n=1}^{\infty} n p_n = \lambda \sum_{n=1}^{\infty} \frac{n p_{n-1}}{\mu(n)} = \lambda \left\{ \sum_{n=1}^{m-1} \left[\frac{n p_{n-1}}{\mu(n)} - \frac{n p_{n-1}}{\mu(m)} \right] + \sum_{n=1}^{\infty} \frac{n p_{n-1}}{\mu(m)} \right\}$$

$$Q = \frac{\lambda}{\mu(m)} \left[1 + Q + \sum_{n=1}^{m-1} n \left[\frac{\mu(m)}{\mu(n)} - 1 \right] p_{n-1} \right]$$

or $$Q = \left(\frac{\rho}{1-\rho} \right) [1+\gamma]$$ for multi-server nodes

$\gamma = 0$ for single server nodes

$$\gamma = \sum_{n=1}^{m-1} (m-n) p_{n-1}$$ useful for numerical computations

$$\rho = \frac{\lambda}{m\mu}$$; Note: need $p_{0}, p_{1}, \ldots, p_{m-2}$ only

- Utilization $$U = \frac{\lambda}{m\mu}$$

Higher Moments for M|M|1 Queue

- Higher Moments of Queue Length

$$E(n^k) = \sum_{n=1}^{\infty} n^k p_n = \rho \sum_{n=1}^{\infty} n^k p_{n-1} = \rho \sum_{n=1}^{\infty} (n-1+1)^k p_{n-1} = \rho \sum_{n=1}^{\infty} \sum_{r=0}^{k} \binom{k}{r} (n-1)^r p_{n-1}$$

$$= \rho \sum_{r=0}^{k} \binom{k}{r} E[n^r]$$

$$\Rightarrow E(n^k) = \frac{\rho}{1-\rho} \{ \sum_{r=0}^{k-1} \binom{k}{r} E[n^r] \}$$

$$k = 1 \Rightarrow Q = \frac{\rho}{1-\rho}$$

$$k = 2 \Rightarrow E(n^2) = \frac{\rho}{1-\rho}(1+2Q) = \frac{\rho(1+\rho)}{(1-\rho)^2} \Rightarrow \sigma_n^2 = \frac{\rho}{(1-\rho)^2} = \frac{Q}{(1-\rho)} \Rightarrow C_n = \frac{\sigma_n}{Q} = \frac{1}{\sqrt{\rho}}$$

$$k = 3 \Rightarrow E(n^3) = \frac{\rho}{1-\rho}[1+3Q+3E(n^2)] = \frac{\rho(1+4\rho+\rho^2)}{(1-\rho)^3}$$

$$\Rightarrow skewness = E[(n-Q)^3] = \frac{\rho(1+\rho)}{(1-\rho)^3} = \frac{E(n^2)}{(1-\rho)}$$

Summary of Lecture 3

- Properties of Exponential and Geometric Distributions
- Properties of the Poisson Process
- Steady-state solutions to DTMC and CTMC
- State Transition Rate Diagrams
- Applications to M/M/1,M/M/1/N,M/M/m,M/M/∞,M/M/m/m Queues
- Performance measures for M/M/1, M/M/∞, and M/M/m Queues