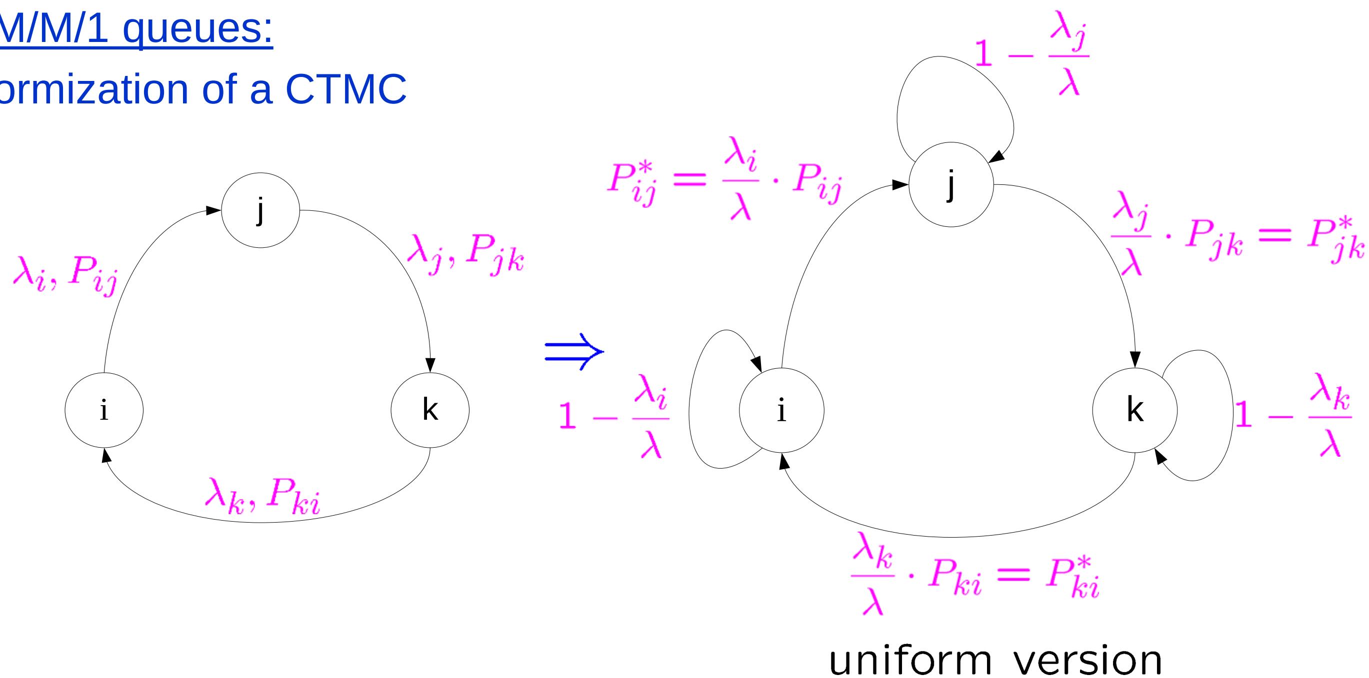
Control of M/M/1 queues

- The stationary distribution of the network state is the product of the marginal distributions at each node $i \Rightarrow$ PRODUCT FORM

  Known as JACKSON'S DECOMPOSITION THEOREM
- Individual nodes behave as if they are M/M/1 queues with rate λv_i and service time per visit is $\dfrac{s_i}{\mu_i(n)}$.

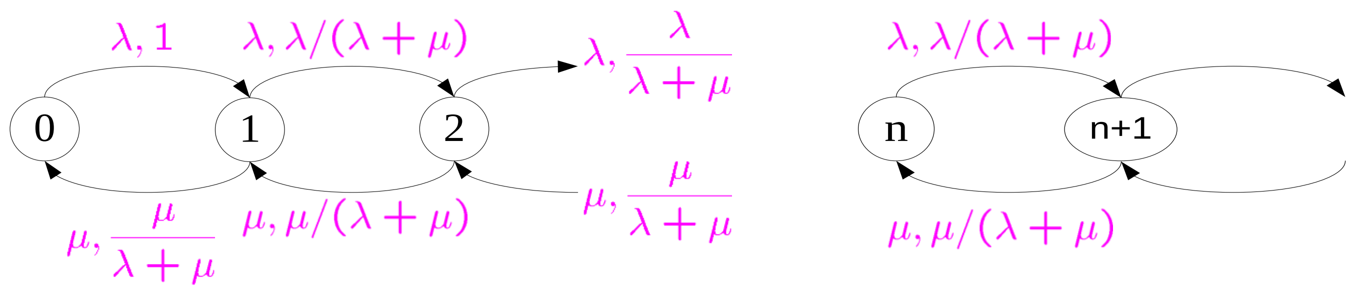
<u>Control of M/M/1 queues:</u>

Recall uniformization of a CTMC



Control of M/M/1 queues

Consider M/M/1 queue with controlled service rate



$$-q_{ii} = \begin{cases} \lambda+\mu & \forall i \geq 1 \\ \lambda \text{ for } i = 0 \end{cases} \qquad p_{ij} = \frac{q_{ij}}{-q_{ii}} = \begin{cases} 1 & i = 1, j = 1 \\ \frac{\lambda}{\lambda+\mu} & j = i+1 \\ \frac{\mu}{\lambda+\mu} & j = i-1;\ i \geq 1 \end{cases}$$

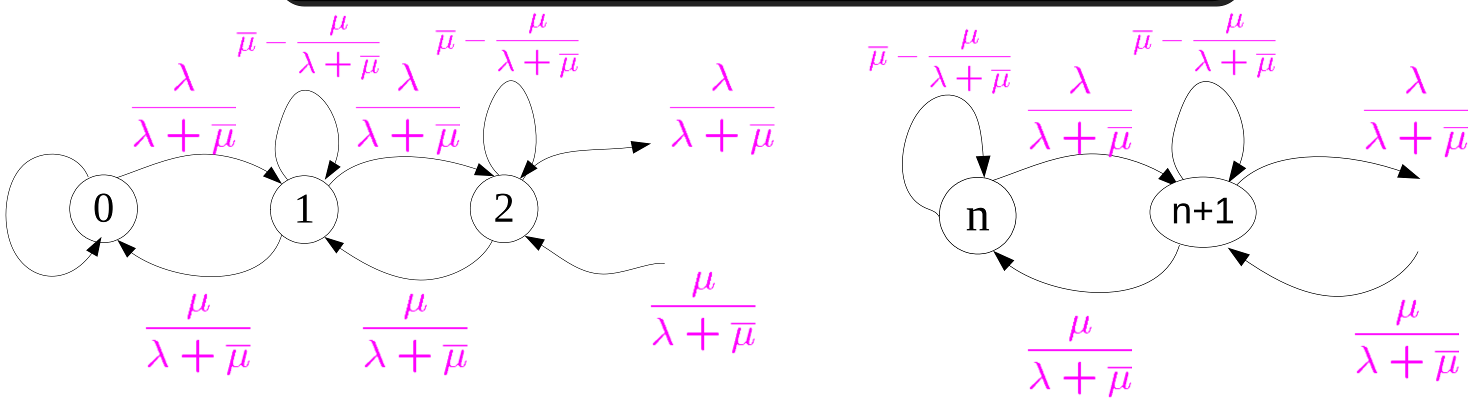
- Suppose that the service rate can selected from a closed subset M of an interval $[0, \overline{\mu}]$
- Service rate μ can be changed at the times when a customer departs from the system (i.e., at the departure epochs).

A good choice of uniformization rate

$$v = \lambda + \overline{\mu}$$

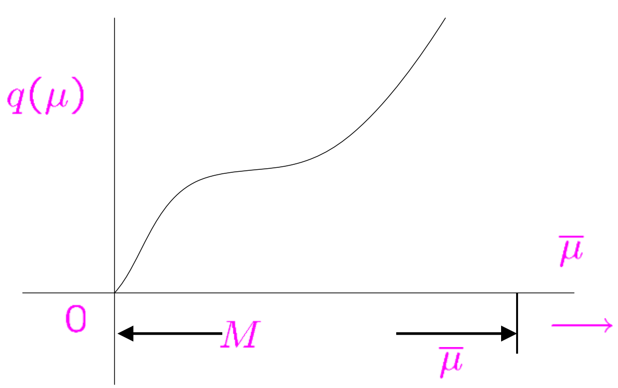
So that the uniformized version is:

Control of M/M/1 queues



- There is a cost $q(\mu)$ per unit time for using rate μ. For example, faster service costs more. Assume $q(0) = 0 \notin q(\mu)$ is continuous

NOT TRUE IN PRACTICE!!
assumption is not necessary

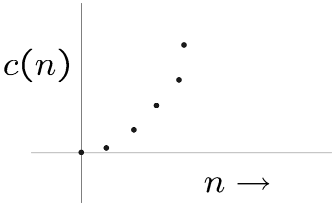


Control of M/M/1 queues

- There is a waiting cost c(n) per unit time when there are n customers in the system (waiting in service or undergoing service). The waiting cost function c(n) is nonnegative, monotonically, nondecreasing, and "convex" in the sense that

$$c(n+2) - c(n+1) \geq c(n+1) - c(n), \; n = 0, 1, 2, \ldots$$

$$\text{or} \quad \frac{c(n+2) + c(n)}{2} \geq c(n+1)$$



Problem: want to minimize the expected discounted cost over an infinite horizon:

$$J_n = E\left\{ \int_0^{\infty} e^{-\beta t} \left[c(X(t)) + q(\mu(t)) \right] dt |_{X(0)=n} \right\}$$

state

Key: the state $X(t)$ and control $\mu(t)$ stay constant between transitions.

Approach: Convert into a discrete-time Markov chain problem

Investigate properties of J

Let

$$t_k = \text{time of occurrence of the } k^{th} \text{ transition} (t_0 = 0 \text{ by convention}$$

$$T_k = t_k - t_{k-1} : \text{ the } k^{th} \text{ transition time interval}$$

$$x_k = x(t_k) : \text{the state after the } k^{th} \text{ transition } [x(t) = x_k \text{ for } t_k \leq t < t_{k+1}]$$

Control of M/M/1 queues

$$u_k = u(t_k) : \text{the control for the } k^{th} \text{ transition } [u(t) = u_k] \text{ for } t_k \le t < t_{k+1}$$

$$\text{So, } J_n = \sum_{k=1}^{\infty} E\left\{ \int_{t_k}^{t_{k+1}} e^{-\beta t}[c(X(t)) + q(\mu(t))]dt|_{X(0)=n} \right\}$$

$$= \sum_{k=0}^{\infty} E\left\{ \int_{t_k}^{t_{k+1}} e^{-\beta t} dt \right\} \cdot E\left\{ c(x_k) + q(\mu_k)|_{X_0=n} \right\}$$

Since the transition time intervals are independent

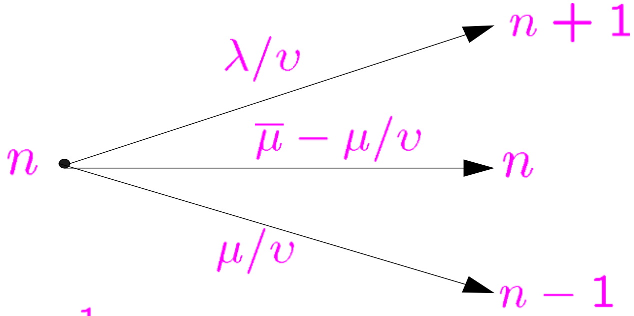
$$E\left[\int_{t_k}^{t_{k+1}} e^{-\beta t} dt \right] = -\frac{E\left\{e^{-\beta t_k}\right\} \cdot \left[1 - E\left\{e^{-\beta T_{k+1}}\right\}\right]}{\beta} = \frac{\alpha^k(1-\alpha)}{\beta}$$

$$\alpha = E\left\{e^{-\beta T}\right\} = \frac{\upsilon}{\upsilon + \beta}$$

So, the expected cost is

$$J_n = \frac{1}{\beta + \upsilon} \sum_{k=1}^{\infty} \alpha^k E\left\{ c(X_k) + q(\mu_k)|_{X_0=n} \right\}$$

Control of M/M/1 queues



$$q(\mu_n) = \begin{cases} q(\mu) & n > 0 \\ 0 & n = 0 \end{cases}$$

$$J_n = \frac{1}{\beta+\upsilon} \min_{\mu \in M} \left\{ c(n) + q(\mu) + \lambda J_{n+1} + (\upsilon - \lambda - \mu) J_n + \mu J_{n-1} \right\};\; n > 1$$

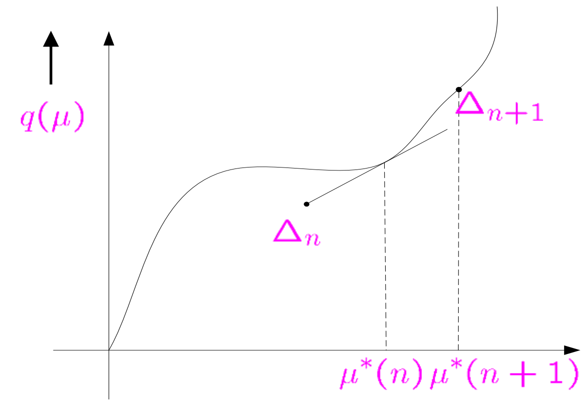
$$J_0 = \frac{1}{\beta+\upsilon} \{ c(0) + \lambda J_1 + (\upsilon - \lambda) J_0 \};\; n = 0$$

An optimal policy is to use at sate I, the service rate that minimizes the expression on the right. So, the optimal policy is to use:

$$\mu_n^* = arg \min_{\mu \in M} \{ q(\mu) - \mu \Delta_n \}$$

Where

$$\Delta_n = J_n - J_{n-1};\;\; n = 1, 2, \ldots$$



Control of M/M/1 queues

Properties of the optimal polity:

1) $\Delta_n \geq \Delta_{n-1}$

2) $\mu_n^* \geq \mu_{n-1}^* \;\Rightarrow\;$ use faster service rate as n $\uparrow$

Proof is based on successive approximation method.

Let $J_n^{(0)} = 0\ \forall n$

For $k = 0, 1, 2, \ldots \mathrm{DO}$

$$J_0^{(k+1)} = \frac{1}{\beta + \upsilon}\left[c(0) + (\upsilon - \lambda)J_0^{(k)} + \lambda J_1^{(k)}\right]$$

$$J_n^{(k+1)} = \frac{1}{\beta + \upsilon}\left[c(0) + q(\mu) + \mu J_{n-1}^{(k)} + (\upsilon - \lambda - \mu)J_n^{(k)} + \lambda J_{n+1}^{(k)}\right];\ n \geq 1$$

Also, let $\Delta_n^{(k)} = J_n^{(K)} - J_{n-1}^{(k)}$

From the theory of Markov decision processes (MDP),

$$\underset{k \to \infty}{lim}\ \Delta_n^{(k)} = \Delta_n;\ \ n = 1, 2, \ldots$$

) it suffices to show that $\Delta_n^{(k)} \geq \Delta_{n-1}^{(k)},\ \forall k$. Proof is by induction. Assume that $\Delta_n^{(k)} \geq \Delta_{n-1}^{(k)}$, we will show that $\Delta_n^{(k+1)} \geq \Delta_{n-1}^{(k+1)}$. By construction $\Delta_n^{(0)} = \Delta_{n-1}^{(0)} = 0$

By definition: $\Delta_{n+1}^{(k+1)} = J_{n+1}^{(k+1)} - J_n^{(k+1)}$

Control of M/M/1 queues

$$\geq \frac{1}{\beta+v}\left\{c(n+1)+q\left[\mu_{n+1}^{k}\right]+\mu_{n+1}^{(k)}J_n^{(k)}+(v-\lambda-\mu_{n+1}^{(k)})J_{n+1}^{(k)}\right.$$
$$+\lambda J_{n+2}^{(k)}-c(n)-q\left[\mu_{n+1}^{(k)}\right]-\mu_{n+1}^{(k)}J_{n-1}^{(k)}$$
$$\left.-(v-\lambda-\mu_{n+1}^{(k)})J_n^{(k)}-\lambda J_{n+1}^{(k)}\right\}$$
$$=\frac{1}{\beta+v}\left\{c(n+1)-c(n)+\lambda\Delta_{n+2}^{(k)}+(v-\lambda)\Delta_{n+1}^{(k)}\right.$$
$$\left.-\mu_{n-1}^{(k)}\left[\Delta_{n+1}^{(k)}-\Delta_n^{(k)}\right]\right\}$$

So, $$(\beta+v)\left[\Delta_{n+1}^{(k+1)}-\Delta_n^{(k+1)}\right] \geq [c(n+1)-2c(n)+c(n-1)]$$
$$+\ \lambda\left[\Delta_{n+2}^{(k)}-\Delta_{n+1}^{(k)}\right]$$
$$+\ \left[v-\lambda-\mu_{n+1}^{(k)}\right]\left[\Delta_{n+1}^{(k)}-\Delta_n^{(k)}\right]$$
$$+\ \mu_{n-1}^{(k)}\left[\Delta_n^{(k)}-\Delta_{n-1}^{(k)}\right]\geq 0$$

M/M/1 Queue with controlled arrival rate: ~ flow control

- $\lambda \in (0,\ \bar{\lambda}) = \Lambda$
  $v = \bar{\lambda}+\mu$

Control of M/M/1 queues

Some equations:

$$J_0 = \frac{1}{\beta + \upsilon} \min_{\lambda \in \Lambda} \left[c(0) + q(\lambda) + (\upsilon - \lambda)J_0 + \lambda J_1\right]$$

$$J_n = \frac{1}{\beta + \upsilon} \min_{\lambda \in \Lambda} \left[c(n) + q(\lambda) + \mu J_{n-1} + (\upsilon - \lambda - \mu)J_n + \lambda J_{n+1}\right]$$

$$\lambda_n^* = \min_{\lambda \in \Lambda} \left[q(\lambda) + \lambda \Delta_{n+1}\right]$$

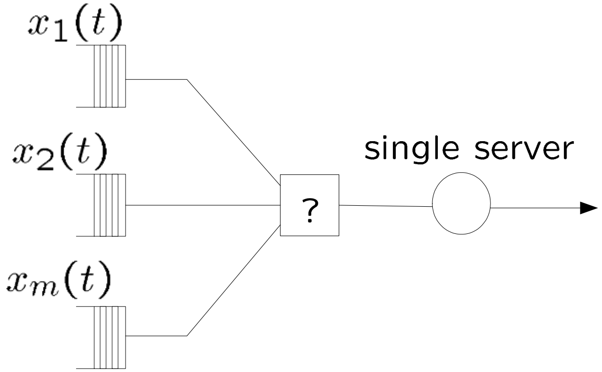
Where $\Delta_n = J_n - J_{n-1}$

Again:

$$\Delta_n \geq \Delta_{n-1}$$

$$\Rightarrow \lambda_n \leq \lambda_{n-1} \text{ or } \lambda_n \downarrow \text{ as n } \uparrow$$

Priority assignment and the μc rule



Control of M/M/1 queues

- m queues sharing a single server. Customers in queue i require service time with mean $1/\mu_i$
- Cost per unit time per customer in queue i, c_i
- Suppose start with ($n_1, n_2, \ldots, n_m$) customers and no further arrivals
- What is the optimal ordering for serving the customers?

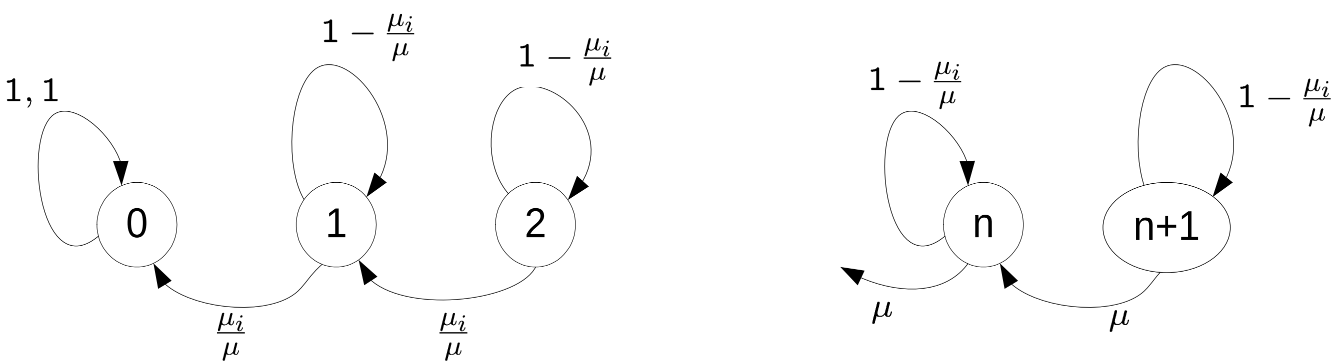
Objective:

$$E\left\{\int_0^\infty e^{-\beta t}\left[\sum_{i=1}^{m} c_i x_i(t)\right] dt\right\}$$

Uniformization:

Let $\mu = \max_i \mu_i$

When queue i is served:



Control of M/M/1 queues

As before: $$\frac{1}{\beta+\mu}\sum_{k=0}^{\infty}\alpha^k E\left\{\sum_{i=1}^{m} c_i X_k^i\right\}$$

X_k^i = # of customers in the i^{th} queue after the k^{th} transition (real or fictitious)

We transform the problem from one of minimizing waiting costs to one of maximizing savings in waiting costs through customer service.

Let

$$i_k = \begin{cases} i & \text{if the } k^{th} \text{ transition corresponds to} \\ & \text{customer departure from queue } i \\ 0 & otherwise \end{cases}$$

Let

$$\begin{aligned} c_{i_0} &= 0 \\ x_0^i &= \text{initial number of customers in queue } i \end{aligned}$$

Control of M/M/1 queues

$$\frac{1}{\beta+\mu}\left[\sum_{i=1}^{m} c_i x_0^i + \sum_{k=1}^{\infty} \alpha^k E\left\{\sum_{i=1}^{m} c_i x_0^i - \sum_{m=0}^{k-1} c_{i_m}\right\}\right]$$

$$= \frac{1}{\beta+\mu}\left[\sum_{k=1}^{\infty} \alpha^k \left(\sum_{i=1}^{m} c_i x_0^i\right) - E\left\{\sum_{m=0}^{\infty} \sum_{k=m+1}^{\infty} \alpha^k c_{i_m}\right\}\right]$$

$$= -\frac{1}{(\beta+\mu)(1-\alpha)} \sum_{i=1}^{m} c_i x_0^i - \frac{\alpha}{(\beta+\mu)(1-\alpha)} E\left[\sum_{k=0}^{\infty} c_{i_k} \alpha^k\right]$$

$$= \frac{1}{\beta} \sum_{i=1}^{m} c_i x_0^i - \frac{\alpha}{\beta} \sum_{k=1}^{\infty} \alpha^k E\left\{c_{i_k}\right\}$$

$$\Rightarrow \quad \max \sum_{k=0}^{\infty} \alpha^k \cdot \underbrace{E\left\{c_{i_k}\right\}}$$

saving in waiting cost rate

Suppose we pick queue i

Customer leaves with prob. $\frac{\mu_i}{\mu}$

does not leave with prob. $\left(1 - \frac{\mu_i}{\mu}\right)$

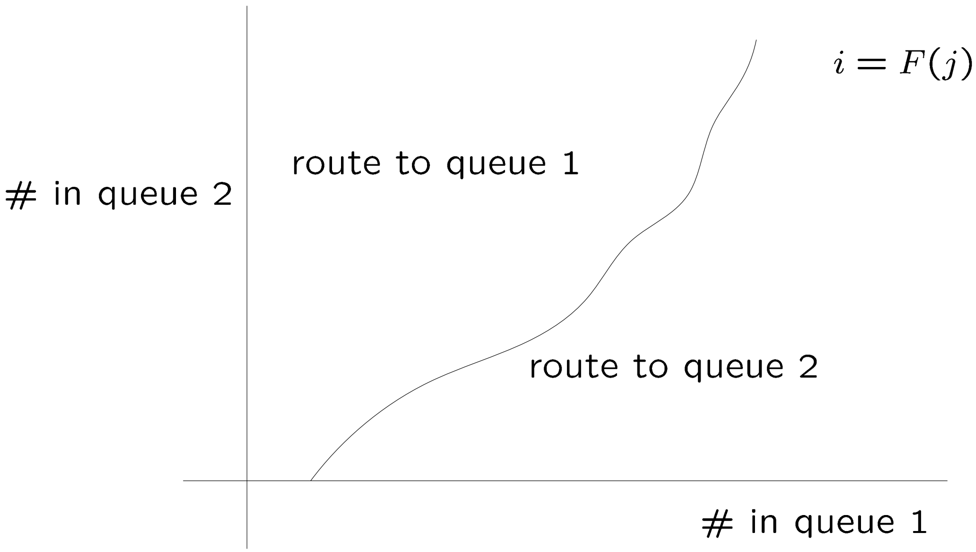
where $x^+ = max(0, x)$

Control of M/M/1 queues

Optimal Policy:

Route an arriving customer to queue 1 iff. the state (i, j) at the time of arrival is 3

$$J(i+1, j) \leq J(i, j+1)$$



Can show this by showing that:

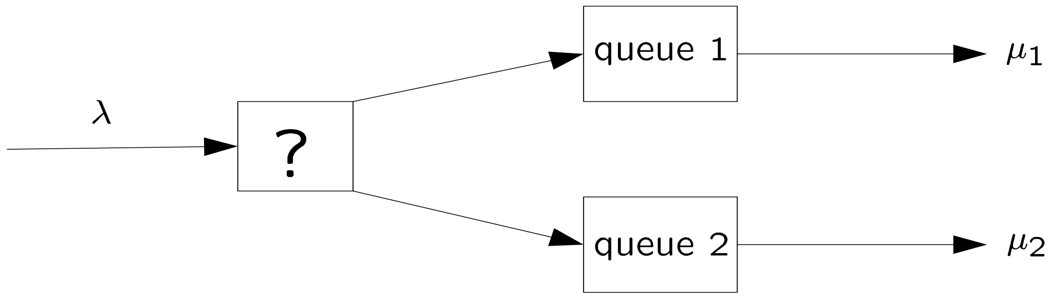
$$\left.\begin{array}{l}\Delta_1(i,j) = J(i+1,j) - J(i,j+1) \\ \Delta_2(i,j) = J(i,j+1) - J(i+1,j)\end{array}\right\} \text{are monotonic nondecreasing}$$

Control of M/M/1 queues

So, expected reward is $\frac{\mu_i}{\mu} \cdot c_i$

It is optimal to serve the nonempty queue i for which $\mu_i c_i$ is maximum.

Threshold policies for routing in a two-node network

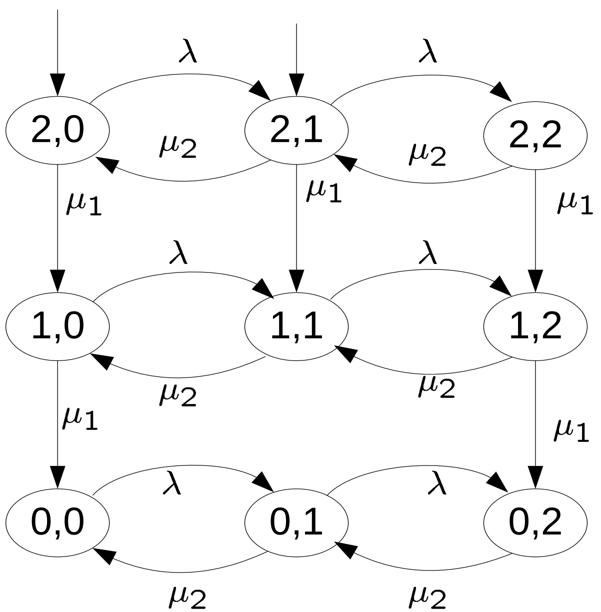


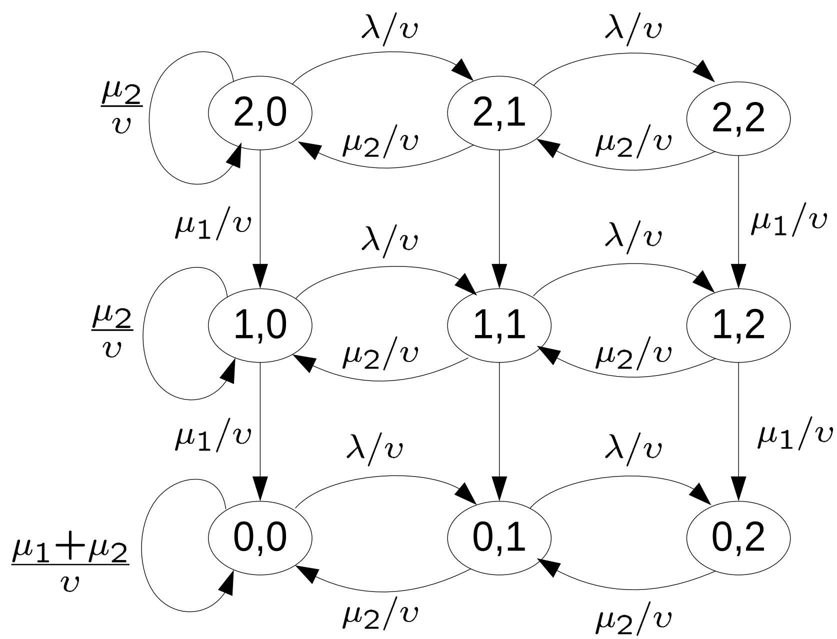
$$\min \; E \int_0^\infty e^{-\beta t} \left[c_1 x_1(t) + c_2 x_2(t) \right] dt$$

Uniform rate:

$$\upsilon = \lambda + \mu_1 + \mu_2$$

Control of M/M/1 queues





$$\upsilon = \lambda + \mu_1 + \mu_2$$

Transition rates when routed to queue 1

$$J(i,j) = \frac{1}{\beta+\upsilon}\left[c_1 i + c_2 j + \mu_1 J((i-1)^+, j) + \mu_2 J(i,(j-1)^+)\right]$$

$$+\frac{\lambda}{\beta+\upsilon}\, min\ [J(i+1,j), J(i,j+1)]$$

Control of M/M/1 queues

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Control of M/M/1 queues

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