The University of Connecticut **Fall 2009**
Dept. of ESE **KRP**

ECE 6437: Computational Methods for Optimization
Problem Set # 1
(Due September 8, 2009)

1. To approximate a function over an interval [0,1] by a polynomial of degree n (or less), we minimize the criterion:

$$f(\underline{a}) = \int_0^1 [g(x) - p(x)]^2 dx$$

where $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$. Find the equations satisfied by the optimal coefficients $\underline{a} = (a_0, a_1, ..., a_n)$.

2. a) Using the first-order necessary conditions, find a minimum point of the function:

$$f(x,y,z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9$$

Verify that the point is a relative minimum by verifying that the second order sufficiency conditions hold. Prove that the point is a global minimum point.

b) Compute the gradient and Hessian of the Rosenbrock's "banana" function:

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

Show that the point (1,1) is a local minimum of this function, and that the Hessian matrix at that point is positive definite.

c) Show that the problem of finding a rectangular parallelepiped of unit volume that has minimum surface area corresponds to the minimization over x >0 and y>0 of

$$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$$

Find the optimal solution and verify that it is a minimum.

3. a) Consider the quadratic cost function:

$$f(\underline{x}) = \frac{1}{2} \underline{x}^T A \underline{x} - \underline{b}^T \underline{x} + c$$

where A is a positive definite matrix. Compute the gradient and Hessian. Show that the optimal solution is given by the set of linear equations:

$$A\underline{x} = \underline{b}$$

Discuss various methods that you know of for solving $A\underline{x} = \underline{b}$ and their computational complexity. What if A is positive semi-definite?

b) This problem illustrates scaling the function to obtain well-conditioned Hessians. Suppose that $g(\underline{z}) = f(\underline{x})$, where $\underline{x} = S\,\underline{z}$ for some nxn matrix S. Show that

$$\nabla g(\underline{z}) = S^T \nabla f(\underline{x}) \; and \; \nabla^2 g(\underline{z}) = S^T \nabla^2 f(\underline{x}) S.$$

Apply this relationship to problem 3 a) when $A = (SS^T)^{-1}$. Interpret $g(\underline{z})$ in this case.

4. We want to find a point $\underline{x}$ in the plane whose sum of weighted distances from a given set of points $\{y_1, y_{2,\ldots,} y_n\}$ is minimized. This point is called the Weber point of the set of points $\{y_1, y_{2,\ldots,} y_n\}$. Mathematically, the problem is:

$$\min imize \sum_{i=1}^{n} w_i \| \underline{x} - \underline{y}_i \|_2^2$$

$$subject\ to\ \underline{x} \in R^n$$

where $w_1, w_{2,\ldots,} w_n$ are given positive scalars. Find the Weber point.

5. Consider the scalar square-root problem:

$$Given\ a > 0, find\ x\ such\ that\ f(x) = x^2 - a = 0.$$

Starting with an initial iterate $x_0 > 0$, the following algorithm computes x_k ($k \geq 1$) as follows:

$$x_k = \frac{1}{2}\left(x_{k-1} + \frac{a}{x_{k-1}}\right)$$

Does the process converge? Define error at iteration k as:

$$e_k = x_k - \sqrt{a}$$

Derive a recursive relation for e_k in terms of e_{k-1}.

6. State whether each of the following functions is convex, concave, or neither.

$$(a)\, f(x) = e^x, (b)\, f(x) = e^{-x}, (c)\, f(x) = x^{-2}, (d)\, f(x) = x + \ln x\ for\ x > 0$$
$$(e)\, f(x) = |\,x\,|, (f)\, f(x) = x \ln x\ \ for\ x > 0,\ (g)\, f(x) = x^{2k}, where\ k\ is\ an\ integer,$$
$$(h)\, f(x) = x^{2k+1}, where\ k\ is\ an\ integer$$

7. (Let Ω be a convex subset of R^n and A be an m by n matrix. Then prove that the set $A\Omega = \{\underline{z} \mid \underline{z} = A\underline{x}, \underline{x} \in \Omega\}\ is\ convex.$

8. Let f_1 and f_2 be convex functions over a convex set Ω (Ω is a convex subset of R^n), and b_1 and b_2 are two nonnegative scalars. Show that $g(\underline{x}) = b_1 f_1(\underline{x}) + b_2 f_2(\underline{x})$ is convex for all $\underline{x} \in \Omega$.