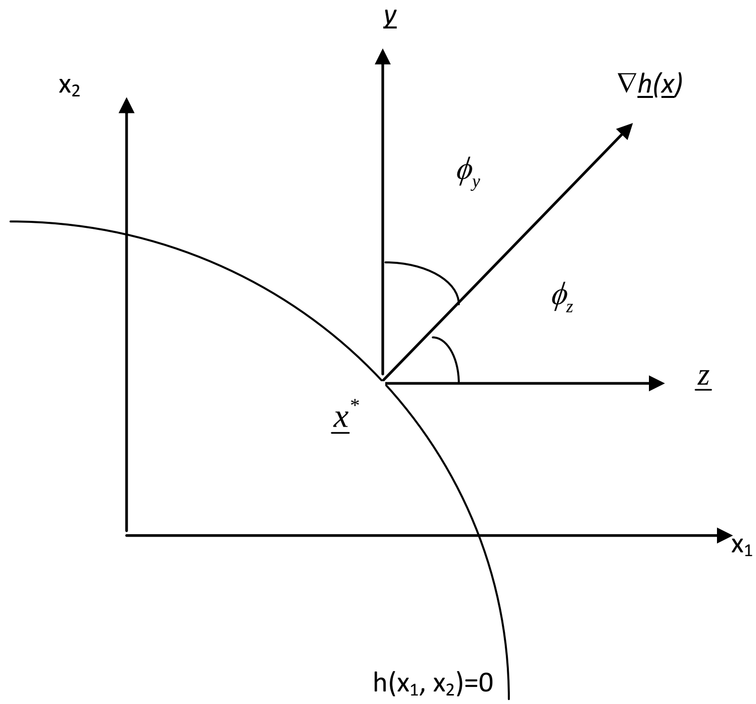
The University of Connecticut **Fall 2009**
Dept. of ECE **KRP**

Problem Set # 7
(Due Date: October 27, 2009)

1. A cardboard box for packing quantities of small foam balls is to be manufactured. The top, bottom, and front faces must be double of weight (i.e., two pieces of cardboard). A problem posed is to find the dimensions of such a box that maximizes the volume for a given amount of cardboard, equal to 72 sq. ft.
    a. What are the first order necessary conditions?
    b. Find the dimensions *x, y, z*
    c. Verify the second order conditions
2. After a heavy military campaign, a certain army requires many new shoes. The quarter master can order three sizes of shoes. Although he does not know precisely how many of each size are required, he feels that the demand for the three sizes is independent and the demand for each size is uniformly distributed between zero and three thousand pairs. He wishes to allocate his shoe budget of four thousand dollars among the three sizes so as to maximize the expected number of men properly shod. Small shoes cost \$1 per pair, medium shoes cost \$2 per pair, and large shoes cost \$4 per pair. How many pairs of each size should he order?
3. An egocentric young man has just inherited a fortune *F* and is now planning how to spend it so as to maximize his total lifetime enjoyment. He deduces that if *x(k)* denotes his capital at the beginning of year *k*, his holdings will be approximately governed by the difference equation
$$x(k+1) = \alpha x(k) - u(k)$$
$$x(0) = F$$
where $\alpha \geq 1$ (with α-1 as the interest rate of investment) and where *u(k)* is the amount spent in year *k*. He decides that the enjoyment achieved in year *k* can be expressed as $\psi(u(k))$ where ψ, his utility function, is a smooth function and that his total lifetime enjoyment is
$$J = \sum_{k=0}^{N} \psi(u(k))\beta^k,$$
where the term β^k, ($0<\beta<1$) reflects the notion that future enjoyment is counted less today. The young man wishes to determine the sequence of expenditures that will maximize his total enjoyment subject to the condition *x(N+1)*=0.
    a. Find the general relationship for the spending sequence for this problem.
    b. Find the solution for the special case $\psi(u)=u^{1/2}$.
4. Consider a smooth curve on the plane described by the equation $h(x_1, x_2) = 0$. Consider also two given points $\underline{y} = (y_1, y_2), \underline{z} = (z_1, z_2)$ which lie in relation to the surface as shown in the figure. Show that if a point $\underline{x}^* = (x_1^*, x_2^*)$ on the curve is such that the sum of the Euclidean distances $\|\underline{y} - \underline{x}\| + \|\underline{z} - \underline{x}\|$ is minimized at $\underline{x}^*$ over all points on the curve, then the angles ϕ_y, ϕ_z shown in the figure must be equal (Fermat's principle in optics).



5. A farmer producing annually x_i units of a certain crop stores (1-u_i) x_i of his production, where $0 \le u_i \le 1$ and invests the remaining $u_i\ x_i$ units thus increasing the next year's production to a level x_{i+1} given by :

$$x_{i+1} = x_i + \alpha u_i x_i, i = 0,1,2,\ldots, N-1$$

where $\alpha \ge 0$ is known. The problem is to find $u_i (0 \le i \le N-1)$ which maximizes the total production stored over N years:

$$x_N + \sum_{i=0}^{N-1} (1-u_i) x_i$$

Show that the optimal control sequence is given by:

a) If $\alpha > 1, u_i^* = 1 \quad (0 \le i \le N-1)$

b) If $0 < \alpha < 1/N, u_i^* = 1 \quad (0 \le i \le N-1)$

c) If $1/N \le \alpha \le 1$, there exists a k' such that $u_i^* = 1 \quad (0 \le i \le N-k'-1)$ and $u_i^* = 0 \quad (N-k' \le i \le N-1)$ and k' is such that $1/(k'+1) < \alpha \le 1/k'$

6. A company has available for sale a quantity of *Q* units of a certain product which will be sold via *n* market outlets. The quantity d_i demanded at outlet *i, i=1,2,…,n*, and the price of sale p_i ,

i=1,2,...,n are known to the company. The company wishes to determine the quantities s_i^* with $0 \le s_i^* \le d_i, i = 1, 2, ..., n$ to be sold at each outlet which maximizes the revenue:

$$\sum_{i=1}^{n} p_i s_i$$

from the sale. Assuming that $p_i > 0, d_i > 0$ and $(d_1 + d_2 + ... + d_n) \ge Q$, show that there exists a cut off price level *y* for each *i* such that if $p_i > y, s_i^* = d_i$ and if $p_i < y$ then $s_i^* = 0$ What happens if $p_i = y$? Describe a procedure for obtaining optimal s_i^*.

7. Derive the dual of the projection problem:

$$\min \| \underline{z} - \underline{x} \|^2 \text{ subjected to } A\underline{x} = \underline{0}$$

where the *m by n* matrix *A* and the vector $\underline{z} \in R^n$ are given. Show that the dual can also be interpreted as a projection problem.

8. Consider the optimal control problem of minimizing

$$\sum_{i=0}^{N-1} \| u_i \|^2$$

Subject to $\underline{x}_{i+1} = A\underline{x}_i + B\underline{u}_i, i = 0, 1, 2, ..., N-1, \underline{x}_0$ is given, and $\underline{x}_N \ge \underline{c}$. Show that the dual problem is of the form :

$$\text{maximize} \underline{\mu}' Q \underline{\mu} + \underline{\mu}' \underline{d} \text{ subjected to } \underline{\mu} \ge \underline{0}$$

where *Q* is an appropriate *n* by *n* matrix (n is the dimension of $\underline{x}$) and $\underline{d}$ is an appropriate *n*-dimensional column vector.