The University of Connecticut **Fall 2009**
Dept. of ESE **KRP**

Problem Set # 8
(Due Date: November 10. 2009)

1. a) One useful application of projection theorem is in the context of gradient projection algorithm. Suppose $\underline{g}$ is the gradient of $f(\underline{x})$. Then, the projected gradient $\underline{p}$ can be computed by solving: minimize $\| \underline{g} - \underline{p} \|_2^2$ subject to $A\underline{p} = \underline{0}$. Using the KKT conditions, solve this problem and derive the formula for the projected gradient.

   b) Show that finding $\underline{d}$ that solves $\min \underline{g}^T \underline{d}$ *subject to* $A\underline{d} = 0, \| \underline{d} \|_2^2 = 1$ gives a vector $\underline{d}$ that has the same direction as the negative projected gradient.

2. Derive the dual of the projection problem

$$\min \| \underline{z} - \underline{x} \|_2^2$$
$$\text{subject to } A\underline{x} = \underline{0}$$

   where the mxn matrix A and the vector $\underline{z}$ are given. Show that the dual problem is also a problem of projection on a subspace.

3. Consider the problem $\min f(\underline{x}) = \frac{1}{2}(x_1^2 - x_2^2) - 3x_2$ *subject to* $x_2 = 0$.

   a. Calculate the optimal solution and the Lagrange multiplier.

   b. For $k=0,1,2$ and $c^k = 10^{k+1}$, calculate and compare the iterates of the quadratic penalty method and the method of multipliers (augmented Lagrangian) with $\lambda^0 = 0$. Interpret the iterates geometrically for both methods. For what values of c would the augmented Lagrangian have a minimum and for what values of c would the method converge?

   c. Verify that the second order method of multipliers converges in a single iteration provided c is sufficiently large, and estimate the threshold value for c.

4. The purpose of this exercise is to show how to treat two-sided inequality constraints by using a single multiplier per constraint. Consider the problem

$$\min f(\underline{x})$$
$$\text{subject to } \alpha_j \le g_j(\underline{x}) \le \beta_j, \; j = 1,2,..,r$$

   The method consists of sequential minimizations of the form

$$\min_{\underline{x}} \Big[f(\underline{x}) + \sum_{i=1}^{r} P_j(g_j(\underline{x}), \mu_j^k, c^k) \Big]$$

$$\text{where } P_j(g_j(\underline{x}), \mu_j^k, c^k) = \min_{u_j \in [g_j(\underline{x}) - \beta_j, g_j(\underline{x}) - \alpha_j]} \Big[\mu_j^k u_j + \frac{c^k}{2} u_j^2 \Big]$$

   Each of these minimizations is followed by the multiplier iteration

$$\mu_j^{k+1} = \begin{cases} \mu_j^k + c^k(g_j(\underline{x}^k) - \beta_j) \; if \; \mu_j^k + c^k(g_j(\underline{x}^k) - \beta_j) > 0 \\ \mu_j^k + c^k(g_j(\underline{x}^k) - \alpha_j) \; if \; \mu_j^k + c^k(g_j(\underline{x}^k) - \alpha_j) < 0 \\ 0 \; otherwise \end{cases}$$

where $\underline{x}^k$ is a minimizing vector. Justify the method by introducing artificial variables u_j by converting the constraint to equivalent form $\alpha_j \le g_j(\underline{x}) - u_j \le \beta_j, u_j = 0, j = 1,2,..,r$ and applying the method of multipliers for this problem, where only the constraints $u_j=0$ are eliminated by means of a quadratic penalty function.

5. Find the sets of all optimal solutions and Lagrange multipliers, and sketch the dual function for the following two-dimensional convex programming problems:

$$a)\; min\; x_1 \; subject\; to \; |x_1| + |x_2| \le 1$$
$$b)\; min\; x_1 \; subject\; to \; |x_1| + |x_2| \le 1, |x_1| \le 1, |x_2| \le 1$$

6. Consider the binary optimization problem (look at constraints on variables)

$$min \; f(\underline{x}) = 10x_1 + 3x_2$$
$$subject\; to \; 4 - 5x_1 - x_2 \le 0, x_1, x_2 = 0 \, or \, 1$$

(a) Sketch the set of constraint-cost pairs $\{(4 - 5x_1 - x_2), 10x_1 + 3x_2 | x_1, x_2 = 0 \textbf{ or } 1\}$

(b) Sketch the dual function (Hint: dualize only the inequality constraint)

(c) Solve the problem and its dual. Relate the solutions to your sketch in part (a)

7. Given objects $i=1,2,..,n$ with positive weights w_i and values v_i, we want to assemble a subset of the objects so that the sum of the weights of the subset does not exceed a given $A > 0$, and the sum of the values of the subset is maximized. The problem is the so-called knapsack problem given by

$$max \sum_{i=1}^{n} v_i x_i \; subject\; to \sum_{i=1}^{n} w_i x_i \le A, x_i \in \{0,1\}, i = 1,2,..,n$$

Form the dual by dualizing the inequality constraint. Use a graphical procedure to solve the dual problem.