Lecture 4: Duality

Prof. Krishna R. Pattipati
Dept. of Electrical and Computer Engineering
University of Connecticut
Contact: krishna@engr.uconn.edu; (860) 486-2890

February 11 & 18, 2016

Outline

- What is duality?
  - Examples
  - Dual of standard and inequality constrained LPs
- Properties
  - Minimum of primal ≡ Maximum of dual
  - Dual of dual ≡ Primal
  - Interpretations as shadow prices
- Application of Duality
  - Game theory
  - Large-scale mathematical programming

What is Duality?

- Q: Is there anything more to LP than revised simplex? Yes!!
- What is duality?
  - Associated with every LP, there exists a dual LP
  - Original LP is called *Primal* LP
  - If Primal LP is one of *minimization*, then Dual LP is one of *maximization*
- Duality occurs in many areas of science and engineering
  - Geometry
    - Minimum distance from the origin to points on a line ≡ maximum distance from the origin to planes through that line
  - Systems Theory
    - Observability ⇔ controllability
    - State ⇔ costate, adjoint state vector, Lagrange multipliers, dual variables
    - LQR ⇔ MMSE estimators
  - Convex programming . . . ECE 6437
  - Philosophy: dualistic versus non-dualistic
  - Voltage-current, force-position, Kirchoff's current and voltage laws

$\underline{\lambda}$ at termination is related to optimal cost

- Duality in LP
  - Consider standard LP (also called primal problem)

$$\min \underline{c}^T \underline{x} \quad s.t.\ A\underline{x} = \underline{b}, \quad \underline{x} \geq \underline{0}$$

  - Transformed problem

$$\min \underline{c}_B^T B^{-1}\underline{b} + \left(\underline{c}_N^T - \underline{c}_B^T B^{-1} N\right)\underline{x}_N$$

$$s.t.\ \underline{x}_B = B^{-1}\underline{b} - B^{-1}N\underline{x}_N \geq \underline{0}$$

$$\underline{x}_N \geq \underline{0}$$

  - Define $\underline{p}^T = \underline{c}_N^T - \underline{c}_B^T B^{-1} N = \underline{c}_N^T - \underline{\lambda}^T N$
  - Optimal if:
    - For non-basic variables $\underline{p}_N^T = \underline{c}_N^T - \underline{\lambda}^T N \geq \underline{0}, \quad \underline{\lambda}^T = \underline{c}_B^T B^{-1}$
    - Also for basic variables $\underline{p}_B^T = \underline{c}_B^T - \underline{c}_B^T B^{-1} B = \underline{0}$

Reduced costs of basic variables are zero

  - So, we obtain the key result:

$\Rightarrow \left(\underline{c}_B^T \mid \underline{c}_N^T\right) - \underline{\lambda}^T [B \mid N] \geq \underline{0}$ or $\underline{c}^T - \underline{\lambda}^T A \geq \underline{0} \Rightarrow \underline{\lambda}^T A \leq \underline{c}^T$

$\Rightarrow$ The simplex multipliers satisfy the constraint $\underline{\lambda}^T A \leq \underline{c}^T$

$\Rightarrow$ Optimal cost $= \underline{c}_B^T B^{-1}\underline{b} = \underline{\lambda}^T \underline{b}$

Dual of SLP

- Suppose we formulate the problem:

$$\max \underline{\lambda}^T \underline{b}$$
$$s.t.\ \underline{\lambda}^T A \le \underline{c}^T$$

- Cannot have minimum since

  $\underline{\lambda} = 0,\quad$ ok

  $\underline{\lambda} = -\infty,$ may be ok; at $\underline{\lambda} = \underline{c}_B^T B^{-1}$ cost of dual = optimal cost of primal

• So, we have our *first result* linking primal and dual:

Primal

$$\min \underline{c}^T \underline{x}$$
$$s.t.\ A\underline{x} = \underline{b}$$
$$\underline{x} \ge \underline{0}$$

Dual

$$\max \underline{\lambda}^T \underline{b}$$
$$s.t.\ \underline{\lambda}^T A \le \underline{c}^T$$

This is because of equality constraint $A\underline{x} = \underline{b}$

- Note that **no restriction on sign of $\underline{\lambda}$ variables**
- **This relation is called asymmetric form of the dual**
- *m* equality constraints $\Leftrightarrow$ *m* variables
- *n* variables $\Leftrightarrow$ *n* inequality constraints
- Roles of $\underline{b}$ and $\underline{c}$ are reversed

Key Questions on Duality

- Example
  - Primal:

$$\min 5x_1 + 4x_2$$
$$s.t.\ x_1 + x_2 = 1$$
$$x_1, x_2 \geq \underline{0}$$

$\Rightarrow$ Optimum at: $x_1 = 0$, $x_2 = 1$

$\Rightarrow$ Optimal cost = 4

  - Dual

$$\max \lambda_1$$
$$s.t.\ \lambda_1 \leq 5,\ \ \lambda_1 \leq 4$$

$\Rightarrow$ Optimum at: $\lambda_1 = 4$

$\Rightarrow$ Optimal cost = 4

- Key questions
  1. Is the minimum of primal = maximum of dual? Yes!!
  2. What happens when you have inequality constraints?
  3. What is the dual of the dual?
  4. What interpretations can we give to dual variables?
  5. How do we solve dual problems?......Dual Simplex
  6. Can we combine primal simplex and dual simplex?...Primal-dual methods

Dual of LP with ≥ inequality constraints

- Let us take questions 2 and 3 first
- ≥ constraints
  - Primal

$$\min \underline{c}^T \underline{x} \quad s.t.\ A\underline{x} \geq \underline{b},\ \underline{x} \geq \underline{0}$$

$$\Rightarrow \quad \min \underline{c}^T \underline{x} + \underline{0}^T \underline{y} \quad s.t.\ A\underline{x} - \underline{y} = (A\ -I)\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix} = \underline{b},\quad \underline{x}, \underline{y} \geq \underline{0}$$

  - Dual

$$\max \underline{\lambda}^T \underline{b} \quad s.t.\ \left(\underline{\lambda}^T A \quad -\underline{\lambda}^T\right) \leq \left(\underline{c}^T \quad \underline{0}^T\right)$$

$$\Rightarrow \underline{\lambda}^T A \leq \underline{c}^T \text{ and } \underline{\lambda} \geq \underline{0}$$

  - So,

Primal

$$\min \underline{c}^T \underline{x} \quad s.t.\ \underline{x} \geq \underline{0},\ A\underline{x} \geq \underline{b}$$

Dual

$$\max \underline{\lambda}^T \underline{b} \quad s.t.\ \underline{\lambda}^T A \leq \underline{c}^T,\ \underline{\lambda} \geq \underline{0}$$

$\Rightarrow \underline{x} \geq \underline{0} \rightarrow \leq \underline{c}^T$ constraints

$\Rightarrow \geq \underline{b} \rightarrow \underline{\lambda} \geq \underline{0}$

$\Rightarrow$ n variables m inequality constraints $\Leftrightarrow$ m variables, n inequality constraints

Dual of LP with ≤ inequality constraints

- $\leq$ constraints

$$\begin{aligned}&\min \underline{c}^T \underline{x}\\ &s.t.\ \underline{x} \geq \underline{0}\\ &\quad A\underline{x} \leq \underline{b}\end{aligned} \quad\longrightarrow\quad \begin{aligned}&\max \underline{\lambda}^T \underline{b}\\ &s.t.\ \underline{\lambda}^T A \leq \underline{c}^T\\ &\quad \underline{\lambda} \leq \underline{0}\end{aligned}$$

- x_j unrestricted

$$\Rightarrow x_j = \bar{x}_j - \hat{x}_j \Rightarrow f = \sum_{\substack{i=1\\ i\neq j}}^{n} c_i x_i + c_j\left(\bar{x}_j - \hat{x}_j\right)$$

$$\underline{b} = \sum_{\substack{i=1\\ i\neq j}}^{n} \underline{a}_i x_i + \underline{a}_j\left(\bar{x}_j - \hat{x}_j\right)$$

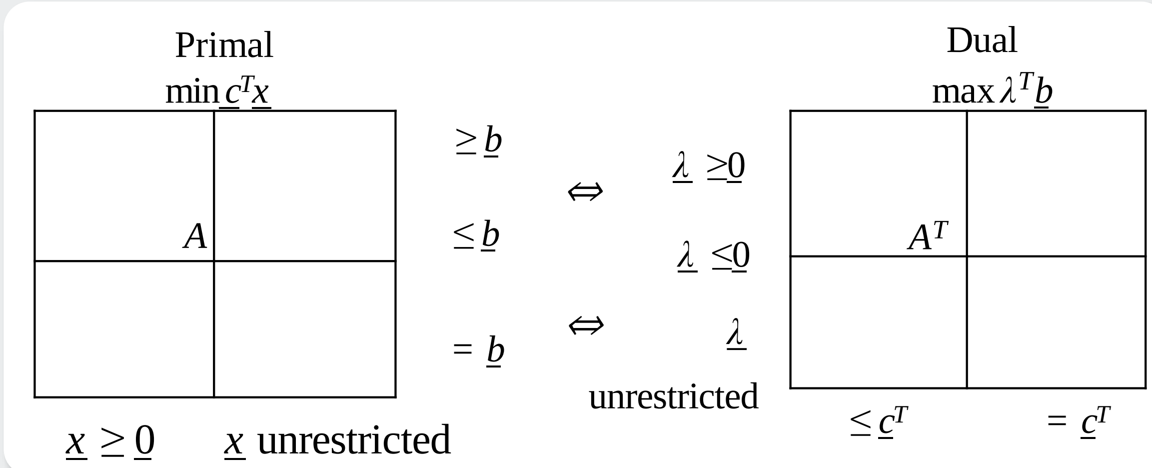
  - Dual

$$\begin{aligned}&\max \underline{\lambda}^T \underline{b}\\ &s.t.\ \underline{\lambda}^T \underline{a}_i \leq c_i,\ \ \forall i \neq j\\ &\quad \underline{\lambda}^T \underline{a}_j = c_j, \text{ since } \underline{\lambda}^T \underline{a}_j \leq c_j \ \&\\ &\quad -\underline{\lambda}^T \underline{a}_j \leq -c_j \Rightarrow \underline{\lambda}^T \underline{a}_j \geq c_j\end{aligned}$$

  - So, **if a variable is unrestricted, the corresponding dual constraint must hold with equality**

Dual of Dual ≡ Primal

- Schematic description of duality



- Dual of a dual ≡ primal (question 3)

$$\begin{aligned} &\max \underline{\lambda}^T \underline{b} \\ &s.t. \quad \underline{\lambda}^T A \leq \underline{c}^T \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\min \; -\underline{\lambda}^T \underline{b} = \underline{\lambda}^T (-\underline{b}) \\ &s.t. \quad \underline{\lambda}^T (-A) \geq -\underline{c}^T \end{aligned}$$

- So $\min \left(\bar{\underline{\lambda}} - \hat{\underline{\lambda}} \right)^T (-\underline{b}) = \min \hat{\underline{\lambda}}^T \underline{b} - \bar{\underline{\lambda}}^T \underline{b}$

$$s.t. \left(-A^T \bar{\underline{\lambda}} + A^T \hat{\underline{\lambda}} \right) \geq -\underline{c}$$

$$\bar{\underline{\lambda}}, \hat{\underline{\lambda}} \geq \underline{0}$$

- Let $\underline{\lambda}_a = \begin{bmatrix} \bar{\underline{\lambda}} \\ \hat{\underline{\lambda}} \end{bmatrix}; \underline{b}_a = \begin{bmatrix} -\underline{b} \\ \underline{b} \end{bmatrix}; A_a^T = [-A^T \;\; A^T]$

Maximum of Dual ≡ Minimum of Primal

- Then

$$\begin{aligned} \min\ & \underline{\lambda}_a^T \underline{b}_a \\ s.t.\ & A_a^T \underline{\lambda}_a \geq -\underline{c} \\ & \underline{\lambda}_a \geq \underline{0} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max\ & -\underline{c}^T \underline{x} \\ s.t.\ & \underline{x}^T A_a^T \leq \underline{b}_a^T \\ & \underline{x} \geq \underline{0} \end{aligned} \quad \Rightarrow \quad \begin{aligned} & \min \underline{c}^T \underline{x} \\ & s.t. -\underline{x}^T A^T \leq -\underline{b}^T \\ & \quad \underline{x}^T A^T \geq \underline{b}^T \\ & \Rightarrow A\underline{x} = \underline{b} \\ & \quad \underline{x} \geq \underline{0} \end{aligned}$$

- Q1: Is maximum of dual ≡ minimum of primal
  - First we prove that maximum of dual ≤ minimum of primal

    ⇒ this is the so-called **weak duality theorem**

  - Recall

Primal $\qquad\qquad$ ***Dual***

$$\begin{aligned} & \min \underline{c}^T \underline{x} \\ & s.t.\ A\underline{x} = \underline{b} \\ & \quad \underline{x} \geq \underline{0} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} & \max \underline{\lambda}^T \underline{b} \\ & s.t.\ \underline{\lambda}^T A \leq \underline{c}^T \end{aligned}$$

Weak Duality Theorem

- Weak duality theorem
  - Suppose $\underline{x}$ and $\underline{\lambda}$ are feasible for primal and dual problems, respectively. Then $\underline{\lambda}^T \underline{b} \leq \underline{c}^T \underline{x}$
  - Proof: $\underline{\lambda}^T \underline{b} = \lambda^T A \underline{x} \leq \underline{c}^T \underline{x}$
  - Since $\underline{x} \geq \underline{0}$, we have $\underline{\lambda}^T A \leq \underline{c}^T$

    $\Rightarrow$ Maximum of dual ≤ minimum of primal

    (or) cost in the dual is never above the cost in the primal

    Dual Cost “gap” **Primal Cost**

    $\Rightarrow$ Fortunately for LP, gap = 0 ⇒ max. dual = min. primal

$$Optimal\ \Pr imal = \underline{c}_B^T \underline{x}_B^* = \underline{c}_B^T B^{-1} \underline{b}$$
$$= \underline{\lambda}^T \underline{b} \leq \underline{\lambda}^{*T} \underline{b}$$
$$But, \underline{\lambda}^T \underline{b} \leq \underline{c}^T \underline{x} \forall feasible\ \underline{x}\ and\ \underline{\lambda}$$
$$so, \underline{\lambda}^{*T} \underline{b} = \underline{c}^T \underline{x}^* = \underline{c}_B^T \underline{x}_B^*$$

- Suppose $\underline{x}$ and $\underline{\lambda}$ are feasible. If $\underline{\lambda}^T \underline{b} = \underline{c}^T \underline{x}$, then $\underline{x}$ and $\underline{\lambda}$ are optimal
  - Proof:
    - No $\underline{\lambda}$ can give a cost greater than $\underline{c}^T \underline{x}$
    - No $\underline{x}$ can give a cost smaller than $\underline{\lambda}^T \underline{b}$ ⇒ must be optimal and gap = 0
  - An LP terminates in one of three ways

    1. Finite optimum, 2. unbounded solution, 3. infeasible solution

Four Primal-Dual Relationships

	Dual: finite	Dual: ∞	Dual: infeasible
Primal: finite	(4)	X	X
Primal: $-\infty$	X	X	(2)
Primal: infeasible	X	(3)	(1)

- Primal finite and dual infeasible case
  - Since primal is finite $\underline{c}_N^T - \underline{\lambda}^T N \geq \underline{0} \Rightarrow \underline{\lambda}^T A \leq \underline{c}^T$

    $\Rightarrow$ A contradiction to the assumption that the dual is infeasible
- Dual finite and primal infeasible case

  $\Rightarrow \max \underline{\lambda}^T \underline{b}$ s.t. $\underline{\lambda}^T A \leq \underline{c}^T$ has finite optimum
  - Convert into SLP as before

$$\begin{aligned} \min \quad & \underline{\lambda}_a^T \underline{b}_a \\ s.t. \quad & \underline{\lambda}_a^T A_a - \underline{y}^T = -\underline{c}^T \\ & \underline{\lambda}_a, \underline{y} \geq \underline{0} \end{aligned}$$

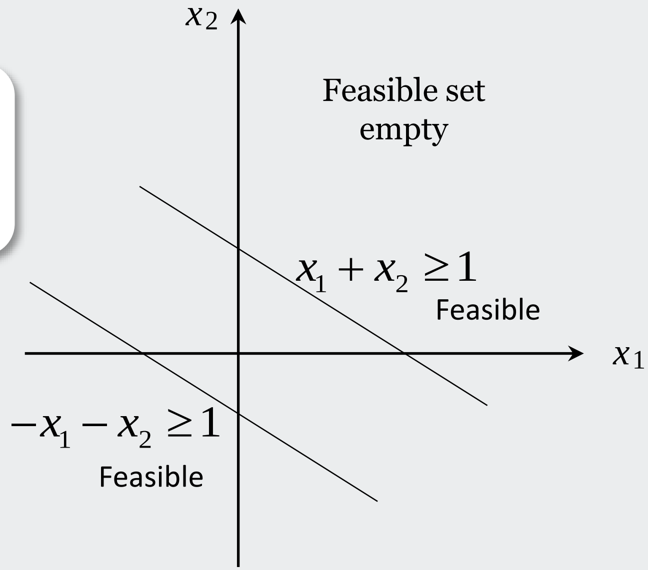
  - Solution finite $\Rightarrow$ by duality $A\underline{x} = \underline{b}$, $\underline{x} \geq \underline{0}$ is feasible since dual of a dual is a primal

Infeasible and unbounded cases

- Case 1: Both feasible sets are empty

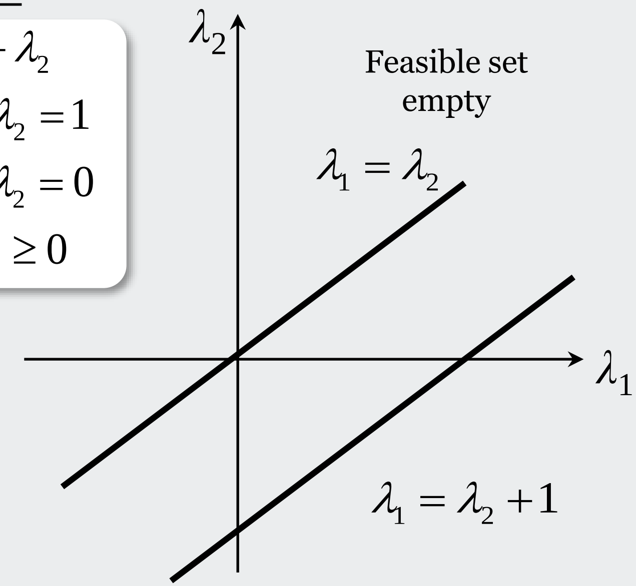
Primal

$$\min x_1$$
$$s.t.\ x_1 + x_2 \geq 1$$
$$-x_1 - x_2 \geq 1$$



Dual

$$\max \lambda_1 + \lambda_2$$
$$s.t.\ \lambda_1 - \lambda_2 = 1$$
$$\lambda_1 - \lambda_2 = 0$$
$$\lambda_1, \lambda_2 \geq 0$$



- Case 2:
  - Minimum in the primal = $-\infty$ (unbounded) $\Rightarrow$ no feasible $\underline{\lambda}$
  - If there is a feasible $\underline{\lambda}$,all feasible costs $c^T \underline{x} \geq \lambda^T \underline{b}$

    $\Rightarrow$ cost cannot go down to $-\infty$
  - Example:

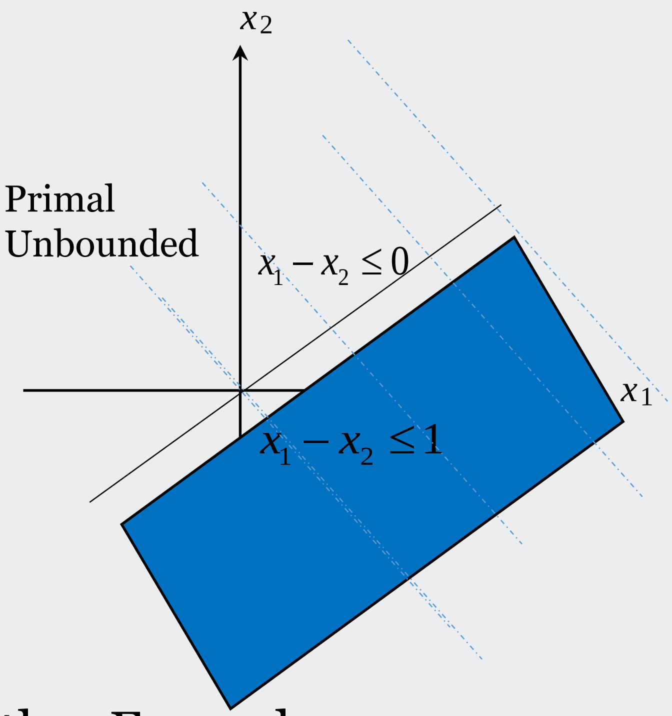
Primal:

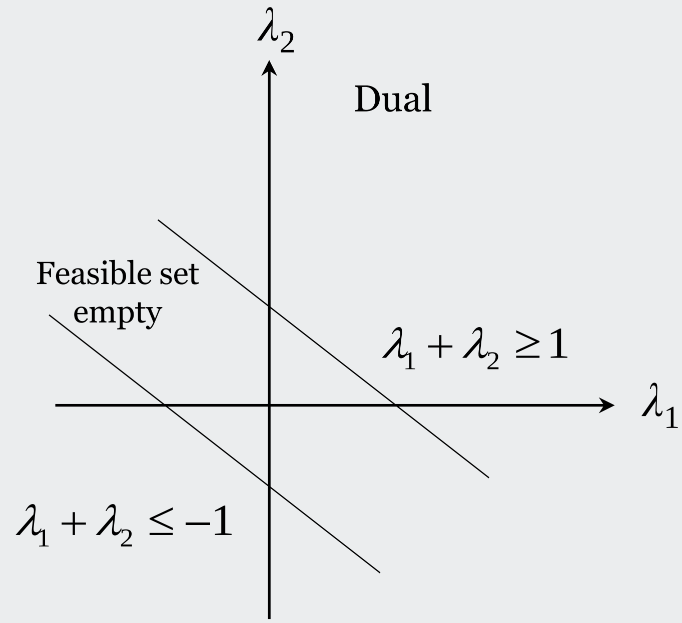
$$\min -(x_1 + x_2)$$
$$s.t.\ x_1 - x_2 \leq 1$$
$$x_1 - x_2 \leq 0$$
$$x_1, x_2 \geq 0 \Rightarrow \text{primal unbounded}$$

Dual:

$$\max \lambda_1$$
$$s.t. -\lambda_1 - \lambda_2 \leq -1$$
$$\lambda_1 + \lambda_2 \leq -1 \Rightarrow \text{dual infeasible}$$

Infeasible dual and finite-finite cases





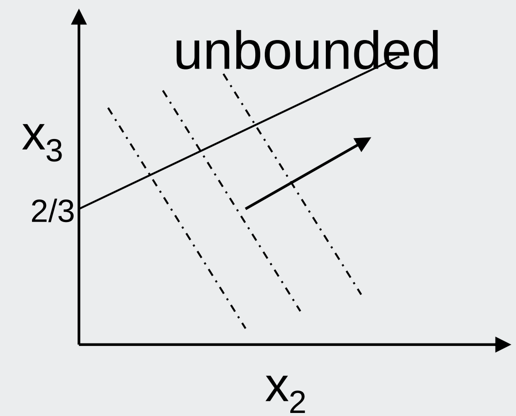
Need a bound on x_2

- Another Example

$$\begin{aligned} &\max\ x_1 + 4x_2 + x_3 \\ &s.t.\ 2x_1 - 2x_2 + x_3 = 4 \\ &\quad x_1 - x_3 = 1 \\ &\quad x_2 \ge 0;\ x_3 \ge 0 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} &\max\ 4x_2 + 2x_3 \\ &s.t. -2x_2 + 3x_3 = 2 \\ &\quad x_2 \ge 0;\ x_3 \ge 0 \end{aligned}$$

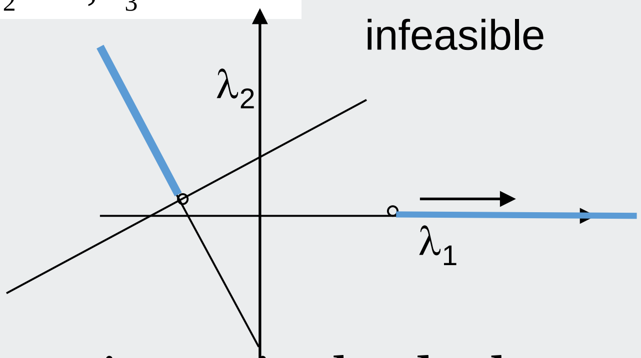


Infeasible dual and finite-finite cases

$$\begin{aligned}&Primal:\\&\max\ x_1+4x_2+x_3\\&s.t.\ 2x_1-2x_2+x_3=4\\&\quad x_1-x_3=1\\&\quad x_2\ge 0;\ x_3\ge 0\end{aligned}$$

$$\begin{aligned}&Dual:\\&\max\ 4\lambda_1+\lambda_2\\&s.t.\ 2\lambda_1+\lambda_2=-1\\&\quad -2\lambda_1\le -4\Rightarrow\lambda_1\ge 2\\&\quad \lambda_1-\lambda_2\le -1\end{aligned}$$

$$\begin{aligned}&Dual:\\&\max\ 2\lambda_1\\&s.t.\ \lambda_1\ge 2\\&\quad \lambda_1\le -2/3\end{aligned}$$



- Case 3: Maximum in the dual = + ∞ ⇒ there is no feasible $\underline{x}$
  - If there is a feasible $\underline{x} \Rightarrow \underline{c}^T\underline{x} \ge \underline{\lambda}^T\underline{b}\ \forall\underline{\lambda}$ …a contradiction ⇒ infeasible $\underline{x}$
- Case 4: Finite-finite case
  - Is finite primal optimal = finite dual optimal
  - Suppose it is: what does it mean?
  - Consider SLP and its dual

Primal

$$\begin{aligned}&\min\ \underline{c}^T\underline{x}\\&s.t.\ \underline{x}\ge\underline{0}\\&\quad A\underline{x}\ge\underline{b}\end{aligned}$$

Dual

$$\begin{aligned}&\max\ \underline{\lambda}^T\underline{b}\\&\text{s.t.}\ \ \underline{\lambda}^T A\le\underline{c}^T\end{aligned}$$

Reduced costs and basic variables at optimum

$$\underline{\lambda}^{T*}\underline{b} = \underline{\lambda}^T A\underline{x}^* = \underline{c}^T\underline{x}^*$$
$$\Rightarrow \left(\underline{\lambda}^{T*}A - \underline{c}^T\right)\underline{x}^* = \underline{0}$$

- But, we know $\underline{x}^* \geq \underline{0}$ and $(\underline{c}^T - \underline{\lambda}^{T*}A) \geq \underline{0}$
- The inner product can be zero in only one way:
- $\underline{x}^*$ must be zero in every component where $(\underline{c}^T - \underline{\lambda}^{T*}A)$ is positive and vice versa $\Rightarrow$ $\underline{x}^*$ and $\underline{\lambda}^*$ must enjoy a special relationship

- **Complementary slackness condition** or **orthogonality condition** or **Karush-Kuhn-Tucker (KKT) conditions**
  - For SLP: feasible vectors $\underline{x}^*$ and $\underline{\lambda}^*$ are optimal iff

$$(\underline{c}^T - \underline{\lambda}^{T*}A)\underline{x}^* = 0$$

  - For each $i = 1,2,\ldots,n$, optimality requires:

1) $\underline{x}_i^* \geq 0 \Rightarrow \underline{\lambda}^{T*}\underline{a}_i = c_i \Rightarrow \text{bfs} \Rightarrow \underline{c}_B^T B^{-1}\underline{a}_i = \underline{c}_B^T \underline{e}_i = c_i$

2) $\underline{x}_i^* = 0 \Leftarrow \underline{\lambda}^{T*}\underline{a}_i < c_i \Rightarrow \text{nonbasic} \Rightarrow c_i - \underline{c}_B^T B^{-1}\underline{a}_i \geq 0$

  - In order to know that we have found an optimal solution $\underline{x}^*$, we must also know the dual solution $\underline{\lambda}^*$

Orthogonality of reduced costs and $\underline{x}^*$

- Example:

$$\min 3x_1 + x_2 + 9x_3 + x_4$$
$$s.t.\ \underline{x} \geq \underline{0}$$
$$x_1 + 2x_3 + x_4 = 4$$
$$x_2 + x_3 - x_4 = 2$$

  - Take basis

$$B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \underline{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = B^{-1}\underline{b} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\underline{\lambda}^T = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

  - Reduced cost vector:

$$p_1 = 3 - \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

$$p_3 = 9 - \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \Rightarrow \text{optimal}$$

$$\Rightarrow \underline{p}^{*T} = \begin{bmatrix} 1 & 0 & 4 & 0 \end{bmatrix}$$

$$\underline{x}^{*T} = [0 \quad 6 \quad 0 \quad 4]$$

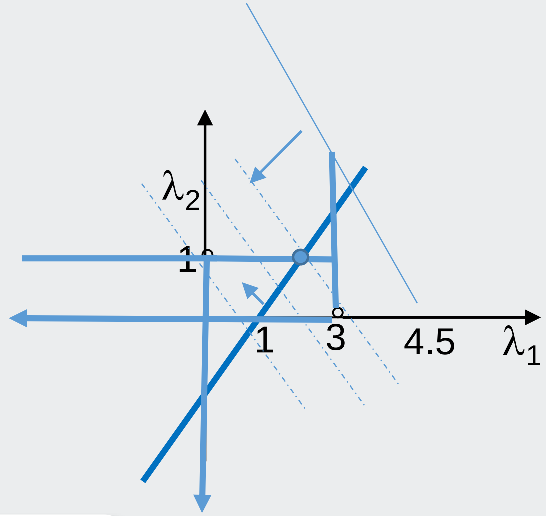
$$\Rightarrow \text{optimal cost} = 10 = \underline{c}^T \underline{x}^* = \underline{\lambda}^{*T} \underline{b}$$

Inequality case

- Dual

$$\max 4\lambda_1 + 2\lambda_2$$
$$s.t.\ \lambda_1 \le 3, \lambda_2 \le 1$$
$$2\lambda_1 + \lambda_2 \le 9$$
$$\lambda_1 - \lambda_2 \le 1$$

optimal $\underline{\lambda}^{T*} = [2 \quad 1]$



$$\underline{c}^T - \underline{\lambda}^{T*} A = [3 \quad 1 \quad 9 \quad 1] - [2 \quad 1]\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} = [1 \quad 0 \quad 4 \quad 0]$$

$$\underline{x}^{T*} = [0 \quad 6 \quad 0 \quad 4]$$

$\Rightarrow$ Inner product of reduced costs and $\underline{x}^*$ is zero and $(\underline{\lambda}^*)^T \underline{b} = 10$

- Case 4: What happens if we had inequality constraints and both primal and dual are finite?
  - Does $(\underline{\lambda}^*)^T \underline{b} = \underline{c}^T \underline{x}^*$ always? Yes!

Primal

$$\min \underline{c}^T \underline{x}$$
$$\text{s.t. } \underline{x} \ge \underline{0}$$
$$A\underline{x} \ge \underline{b}$$

Dual

$$\max \underline{\lambda}^T \underline{b}$$
$$\text{s.t. } \underline{\lambda}^T A \le \underline{c}^T$$
$$\underline{\lambda} \ge \underline{0}$$

  - This is **symmetric form of the dual**
  - Easy to show the complimentary slackness condition

Complementary Slackness Conditions

- In standard form

Primal

$$\min [\underline{c}^T \quad 0]\begin{bmatrix}\underline{x}\\ \underline{y}\end{bmatrix}$$
$$s.t.\ A\underline{x} - \underline{y} = \underline{b}$$
$$\underline{x}, \underline{y} \geq \underline{0}$$

Dual

$$\max \underline{\lambda}^T \underline{b}$$
$$s.t.\ \underline{\lambda}^T (A - I) \leq [\underline{c}^T \quad 0]$$

$Primal:$
$x_1 =$ number of barrels of light crude
$x_2 =$ number of barrels of heavy crude
$\min 56x_1 + 50x_2$
$s.t.\ 0.3x_1 + 0.3x_2 \geq 900,000$
$0.2x_1 + 0.4x_2 \geq 800,000$
$0.3x_1 + 0.2x_2 \geq 500,000$
$x_1 \geq 0; x_2 \geq 0$
$optimal\ point: (0, 3M)$
$Cost: \$150M$

$Dual:$
$\max 100,000[9\lambda_1 + 8\lambda_2 + 5\lambda_3]$
$s.t.\ 0.3\lambda_1 + 0.2\lambda_2 + 0.3\lambda_3 \leq 56$
$0.3\lambda_1 + 0.4\lambda_2 + 0.2\lambda_3 \leq 50$
$s.t.\ \lambda_1 \geq 0; \lambda_2 \geq 0; \lambda_3 \geq 0$
$optimal\ point: (500/3 \ \ 0 \ \ 0)$
$Cost: \$150M$

- Apply c. s. conditions of SLP

$$(\underline{c}^T - \underline{\lambda}^T A)\underline{x} + \underline{\lambda}^T \underline{y} = 0 \Rightarrow (\underline{c}^T - \underline{\lambda}^T A)\underline{x} + \underline{\lambda}^T (A\underline{x} - \underline{b}) = 0$$
$$\Rightarrow (\underline{c}^T - \underline{\lambda}^T A)\underline{x} = 0 \text{ and } \underline{\lambda}^T (A\underline{x} - \underline{b}) = 0$$

- In words,

1) $x_i > 0 \Rightarrow \underline{\lambda}^T \underline{a}_i = c_i$ (basic)
2) $x_i = 0 \Leftarrow \underline{\lambda}^T \underline{a}_i < c_i$ (nonbasic)
3) $\lambda_i > 0 \Rightarrow \underline{a}^i \underline{x} = b$ (nonbasic surplus)
4) $\lambda_i = 0 \Leftarrow \underline{a}^i \underline{x} > b$ (basic surplus)

$x_1 = 0 \Rightarrow \frac{500}{3}(0.3) = 50 < c_1 = 56$
$x_2 > 0 \Rightarrow \frac{500}{3}(0.3) = 50 = c_2$
$\lambda_1 > 0 \Rightarrow 0.3*(0) + 0.3*3M = 0.9M$
$\lambda_2 = 0 \Rightarrow 0.2*(0) + 0.4*3M = 1.2M > 0.8M$
$\lambda_3 = 0 \Rightarrow 0.3*(0) + 0.2*3M = 0.6M > 0.5M$

where $\underline{a}^i$ is row i of A

- We will provide physical interpretations later

Duality Theorem

- Duality Theorem
  - If there is an optimal solution $\underline{x}^*$ for the primal problem, then there is an optimal $\underline{\lambda}^*$ in the dual and the minimum primal cost $\underline{c}^T\underline{x}^*$ = the maximum dual cost $\underline{\lambda}^{T*}\underline{b}$
  - Proof:
    - $\underline{x}^*$ optimal ⇒ $(n-m)$ components are zero and m components are nonnegative

$$\underline{x}^* = \begin{bmatrix} \underline{x}_B^* \\ \underline{x}_N^* \end{bmatrix} = \begin{bmatrix} \underline{x}_B^* \\ 0 \end{bmatrix} \text{ and } \underline{x}_B^* = B^{*-1}\underline{b}$$

    - We know $\underline{c}^T\underline{x}^* = \underline{c}_B^T B^{*-1}\underline{b}$ and $\underline{p}^T = \underline{c}_N^T - \underline{c}_B^T B^{*-1} N^* \geq \underline{0}$
    - Pick $\underline{\lambda}^{*T} = \underline{c}_B^T B^{*-1} \Rightarrow \underline{\lambda}^{*T}\underline{b} = \underline{c}_B^T \underline{x}_B^* = \underline{c}^T\underline{x}^*$
    - In addition: $\underline{\lambda}^T A \leq \underline{c}^T$ from $\underline{p}^T \geq \underline{0} \Rightarrow \underline{\lambda}^*$ is feasible

$$\underline{\lambda}^T A = \underline{c}_B^T B^{*-1}[B^* \quad N^*] = [\underline{c}_B^T \quad \underline{c}_B^T B^{*-1} N^*] \leq \underline{c}^T$$

      ⇒ max. of dual and min. of primal have met
    - Since the dual of the dual =primal, the theorem also says that if the dual has a finite optimal solution, so does the primal
    - Simplex multipliers at the optimum $\underline{x}^*$ solve the dual LP

Dual variables as synthetic prices

- Interpretation of *simplex multipliers* as synthetic prices of unit vectors in R^m (also called shadow prices)
  - $A=[\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_n]$
  - Cost of vector $i = c_i$ since r.h.s $\underline{b}=\sum_{i=1}^{n}\underline{a}_i x_i$ and cost $f=\underline{c}^T\underline{x}$

$$\underline{e}_i = i^{th}\text{unit vector in } R^m \Rightarrow \underline{a}_i=\sum_{i=1}^{m} a_{ij}\underline{e}_i$$

  - If $\underline{a}_i$ is in basis, it costs c_i units per unit of x_i
  - Suppose basis is first m columns and independent
  - What is the cost of $\underline{e}_j$, the j^{th} unit vector

$$\underline{e}_j=\sum_{i=1}^{m}\alpha_i\underline{a}_i \Rightarrow \underline{e}_j=B\underline{\alpha}\Rightarrow\underline{\alpha}=B^{-1}\underline{e}_j=\left(B^{-1}\right)_j\text{; the } j^{th}\text{col. of } B^{-1}$$

$$\text{cost of } \underline{e}_j=\sum_{i=1}^{m}\alpha_i c_i=\underline{c}_B^T\underline{\alpha}=\underline{c}_B^T(B^{-1})_j=\lambda_j$$

  - **Simplex multiplier λ_j is the synthetic price of unit vector $\underline{e}_j$**
- What are the uses of multipliers?
- Pricing out a vector
  - Consider any vector $\underline{a}_k$; Synthetic price of $\underline{a}_k$
  - True cost of $\underline{a}_k=\underline{c}_k$; synthetic price of $\sum_{i=1}^{m}a_{ik}\underline{e}_i=\sum_{i=1}^{m}a_{ik}\underline{\lambda}_i=\underline{\lambda}^T\underline{a}_k$
  - Relative cost=$c_k-\underline{\lambda}^T\underline{a}_k=p_k$ pricing out a vector
  - Optimality ⇒ synthetic price < actual (true) price for a non-basic column $\underline{a}_k$

$$\underline{\lambda}^{*T}=\begin{bmatrix}2 & 1\end{bmatrix}$$

$$\text{cost of }\begin{bmatrix}1\\-1\end{bmatrix}=\lambda_1-\lambda_2=1$$

Dual variables and sensitivity

- Fundamental data items in LP: ($\underline{c}$, A, $\underline{b}$)
- Changes in $\underline{c}$
  - Changes in *non-basic* coefficients
  - Changes in *basic* coefficients
- Changes in $\underline{b}$
- Changing the column of a non-basic variable, i.e., change in A
- Adding a new variable $\Rightarrow$ add a coefficient to $\underline{c}$ and a column in A corresponding to the new variable
  - Change the number of columns in A and size of $\underline{c}$ vector
- Adding a new constraint, i.e., add a coefficient to $\underline{b}$ and a row in A
- What if multiple parameters change?

Changes in objective function coefficients

- Allowable changes in *non-basic variables (NBV)* w/o changing the basis
  - Consider the example again: p_1=1 can change c_1 from 3 to 2 w/o changing basis and optimal solution $\Rightarrow c_1 \to c_1 + \delta_1$ where $\delta_1 \geq -1\,(or)\; 2 \leq c_1 \leq \infty$
  - p_3 = 4 can change c_3 from 9 to 5 w/o changing basis and the optimal solution

$$\Rightarrow c_3 \to c_3 + \delta_3 \text{ where } \delta_3 \geq -4 \Rightarrow\; 5 \leq c_3 \leq \infty$$

  - In general, for non-basic variables

$$\underline{\lambda}^T \underline{a}_i = \underline{c}_B^T B^{-1} \underline{a}_i \leq c_i \leq \infty; i \in NBV$$

f^* does not change
$\underline{x}^*$ does not change
$\underline{\lambda}^*$ does not change

$$\min 3x_1 + x_2 + 9x_3 + x_4$$
$$s.t.\ \underline{x} \geq \underline{0}$$
$$x_1 + 2x_3 + x_4 = 4$$
$$x_2 + x_3 - x_4 = 2$$
$$Basic: x_2, x_4$$
$$\underline{\lambda}^{T*} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$
$$\underline{p}^{*T} = \begin{bmatrix} 1 & 0 & 4 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$
$$B^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
$$j = 2; l = 1$$
$$\delta_2 \leq \min\{1, 4/3\} = 1$$
$$\delta_4 \leq \min\{1, 4/2\} = 1$$

- What if the changes in *NBV* are outside of allowable range?
  - Reduced cost, p_i < 0 and the current basis is no longer optimal
  - Bring x_i into the basis (**good to use primal simplex!**)
- What if the objective function coefficient of a *basic variable* (*BV*) changes (**again good to use primal simplex!**)
  - If c_j is the cost co-efficient of l^{th} basic variable, that is, *j=BV(l)*

then $\underline{c}_B = \underline{c}_B + \delta_j \underline{e}_l$

$$\Rightarrow (\underline{c}_B + \delta_j \underline{e}_l)^T B^{-1} \underline{a}_i = c_i \quad \forall i \in BV \;\&\; (\underline{c}_B + \delta_j \underline{e}_l)^T B^{-1} \underline{a}_i \leq c_i \quad \forall i \in NBV$$

$$\Rightarrow \delta_j (B^{-1}\underline{a}_i)_l \leq p_i \quad \forall i \in NBV \Rightarrow \max_{i \in NBV:(B^{-1}\underline{a}_i)_l < 0} \left(\frac{p_i}{(B^{-1}\underline{a}_i)_l} \right) \leq \delta_j \leq \min_{i \in NBV:(B^{-1}\underline{a}_i)_l > 0} \left(\frac{p_i}{(B^{-1}\underline{a}_i)_l} \right)$$

f^* changes
$\underline{x}^*$ does not change
$\underline{\lambda}^*$ changes

Changes in objective variable coefficients

- Example

$$\min -60x_1 - 30x_2 - 20x_3$$
$$8x_1 + 6x_2 + x_3 + s_1 = 48$$
$$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$$
$$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$$
$$x_i \geq 0; i = 1,2,3$$
$$\underline{c}^T = [-60 \; -30 \; -20 \; 0 \; 0 \; 0]$$

$$Optimal\ BV: x_1, x_3, s_1$$
$$B = \begin{bmatrix} 8 & 1 & 1 \\ 4 & 1.5 & 0 \\ 2 & 0.5 & 0 \end{bmatrix}; B^{-1} = \begin{bmatrix} 0 & -0.5 & 1.5 \\ 0 & 2 & -4 \\ 1 & 2 & -8 \end{bmatrix}$$
$$\underline{x}_B^{*T} = [2 \quad 8 \quad 24]; f^* = c_B^T \underline{x}_B^* = -60*2 - 20*8 + 24*0 = -280$$
$$\underline{\lambda}^{*T} = [-60 \quad -20 \quad 0] B^{-1} = [0 \quad -10 \quad -10]$$
$$f^* = \underline{\lambda}^{*T} \underline{b} = 0*48 - 10*20 - 10*8 = -280$$
$$\underline{p}^{*T} = \underline{c}^T - \lambda^{*T} A = [0\,5\,0\,0\,10\,10]$$

Why are Dual variables Negative?

Allowable ranges for NBV

$$NBV: x_2, s_2, s_3$$
$$x_2: -35 \leq c_2 \leq \infty$$
$$s_2: -10 \leq c_5 \leq \infty$$
$$s_3: -10 \leq c_6 \leq \infty$$

Allowable ranges for BV

$$BV: x_1, x_3, s_1; NBV: x_3, s_2, s_3$$
$$\max_{i \in NBV:(B^{-1}\underline{a}_i)_l < 0} \left(\frac{p_i}{(B^{-1}\underline{a}_i)_l} \right) \leq \delta_j \leq \min_{i \in NBV:(B^{-1}\underline{a}_i)_l > 0} \left(\frac{p_i}{(B^{-1}\underline{a}_i)_l} \right)$$
$$B^{-1}N = \begin{bmatrix} 1.25 & -0.5 & 1.5 \\ -2 & 2 & -4 \\ -2 & 2 & -8 \end{bmatrix}$$

Look at each row for each BV

$$x_1: -60 - \frac{10}{0.5} = -80 \leq c_1 \leq -60 + \min(\frac{5}{1.25}, \frac{10}{1.5}) = -56$$
$$x_3: -20 + \max(\frac{-5}{2}, \frac{-10}{4}) = -22.5 \leq c_3 \leq -20 + \frac{10}{2} = -15$$
$$s_1: 0 + \max(\frac{-5}{2}, \frac{-10}{8}) = -1.25 \leq c_4 \leq 0 + \frac{10}{2} = 5$$

Changes in RHS of constraints

- Sensitivity analysis
  - How does optimal cost change as we change $\underline{b}$ by "a small amount"?
  - Recall that $\partial f / \partial b_i = \lambda_i$ = marginal cost
  - Δb_i ="small" in the sense that the basis does not change
  - So if

$$\underline{b} \rightarrow \underline{b} + \Delta\underline{b} \Rightarrow f^* = (\underline{\lambda}^T)^*(\underline{b}) = \underline{c}^T \underline{x}^* \rightarrow f^* + \Delta f = (\underline{\lambda}^T)^*(\underline{b} + \Delta\underline{b})$$

$$\Delta f = (\underline{\lambda}^T)^* \Delta\underline{b} \Rightarrow \lambda_j = \frac{\Delta f}{\Delta b_j} = \frac{\text{(change in solution)}}{\text{(change in constraint data)}}$$

  - Another way: changes in $\underline{b}$ causes changes in bfs

$$\Rightarrow \underline{x}_B \rightarrow \underline{x}_B + \Delta\underline{x}_B \text{ where } \Delta\underline{x}_B = B^{-1}\Delta\underline{b}$$
$$\Rightarrow \Delta f = \underline{c}_B^T \Delta\underline{x}_B = \underline{c}_B^T B^{-1}\Delta\underline{b} = (\underline{\lambda}^T)^* \Delta\underline{b}$$

  - If $\Delta\underline{b} = \delta\underline{e}_i$, that is, $b_i = b_i + \delta$,

$$\Delta\underline{x}_B = \delta B^{-1}\underline{e}_i = \delta(B^{-1})_i, (B^{-1})_i = i^{th} \text{ column of } B^{-1}$$
$$Need: \underline{x}_B + \delta(B^{-1})_i \geq \underline{0}$$
$$\Rightarrow \Delta f = \underline{c}_B^T \Delta\underline{x}_B = \delta\underline{c}_B^T (B^{-1})_i = \delta(\underline{\lambda}^T)^* \underline{e}_i = \delta\lambda_i$$

Good to work with dual simplex if $\underline{b}$ changes: Lecture 5

$$\min 3x_1 + x_2 + 9x_3 + x_4$$
$$s.t.\ \underline{x} \geq \underline{0}$$
$$x_1 + 2x_3 + x_4 = 4$$
$$x_2 + x_3 - x_4 = 2$$
$$Basic: x_2, x_4$$
$$\underline{x}_B^{*T} = [6 \quad 4]$$
$$\underline{\lambda}^{T*} = [2 \quad 1]$$
$$\underline{p}^{*T} = [1 \quad 0 \quad 4 \quad 0]$$
$$B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$
$$B^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

f^* *changes*
$\underline{x}^*$ *changes*
$\underline{\lambda}^*$ *does not change*

How much can you change b_i

$$\min 3x_1 + x_2 + 9x_3 + x_4$$
$$s.t.\ \underline{x} \geq \underline{0}$$
$$x_1 + 2x_3 + x_4 = 4$$
$$x_2 + x_3 - x_4 = 2$$

$$\underline{\lambda}^T = [1 \quad 1]\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{B^{-1}} = [2 \quad 1]$$

$$p_1 = 3 - [2 \quad 1]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$
$$p_3 = 9 - [2 \quad 1]\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \Rightarrow \text{optimal}$$
$$\Rightarrow \underline{p}^* = [1 \quad 0 \quad 4 \quad 0]$$
$$\underline{x}^* = [0 \quad 6 \quad 0 \quad 4]$$
$$\Rightarrow \text{optimal cost} = 10 = \underline{c}^T \underline{x}^* = \underline{\lambda}^{*T} \underline{b}$$

- A key question often asked is:
  - How much $\Delta \underline{b}$ can we tolerate w/o changing basis:
  - Recall
    $$\underline{x}_B = B^{-1}\underline{b} - B^{-1}N\underline{x}_N \geq \underline{0}$$
    $$\underline{x}_N = \underline{0} \Rightarrow \underline{x}_B = B^{-1}\underline{b}$$
  - Suppose $b_i \to b_i + \delta \Rightarrow \underline{b} = \underline{b} + \delta \underline{e}_i$
  - For feasibility, need $B^{-1}(\underline{b} + \delta \underline{e}_i) \geq \underline{0}$
  - Let $\underline{g} = B^{-1}\underline{e}_i \Rightarrow (B^{-1})_i$ is i^{th} column of B^{-1}
  - Or $\underline{x}_B + \delta \underline{g} \geq \underline{0}$ or $x_{B(j)} + \delta g_j \geq \underline{0},\ j = 1, 2, \ldots, m$

  Equivalently, $\max\limits_{\{j: g_j > 0\}} \left(-\dfrac{x_{B(j)}}{g_j} \right) \leq \delta \leq \min\limits_{\{j: g_j < 0\}} \left(-\dfrac{x_{B(j)}}{g_j} \right)$

- Example: a) $b_f \to b_1 + \delta \Rightarrow$ can find δ when feasibility of $\underline{x}_B$ is violated

$$\underline{g} = B^{-1}\underline{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \max(-6, -4) \leq \delta \Rightarrow -4 \leq \delta \Rightarrow 0 \leq b_1 \leq \infty$$

$$b_2 \to b_2 + \delta, \underline{g} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow -6 \leq \delta \Rightarrow -4 \leq b_2 \leq \infty$$

Other interesting changes

- Changes to a non-basic column ("pricing out a new column")
  - Cost c_i; column $\underline{a}_i$

    $If\ \ p_i = c_i - \underline{\lambda}^{*T}\underline{a}_i \geq 0,$ basis is still the same.

    Otherwise, bring variable x_i into the basis.
- Adding a new variable is similar to changing a non-basic column
- What if multiple cost coefficients are changed?
  - For non-basic, reduced costs tell us whether the basis is optimal or not
  - For multiple changes in basic coefficients, use 100% rule

$c_j =$ original cost coefficient with bounds $c_j - D_j \leq c_j \leq c_j + I_j; D_j \geq 0; I_j \geq 0$

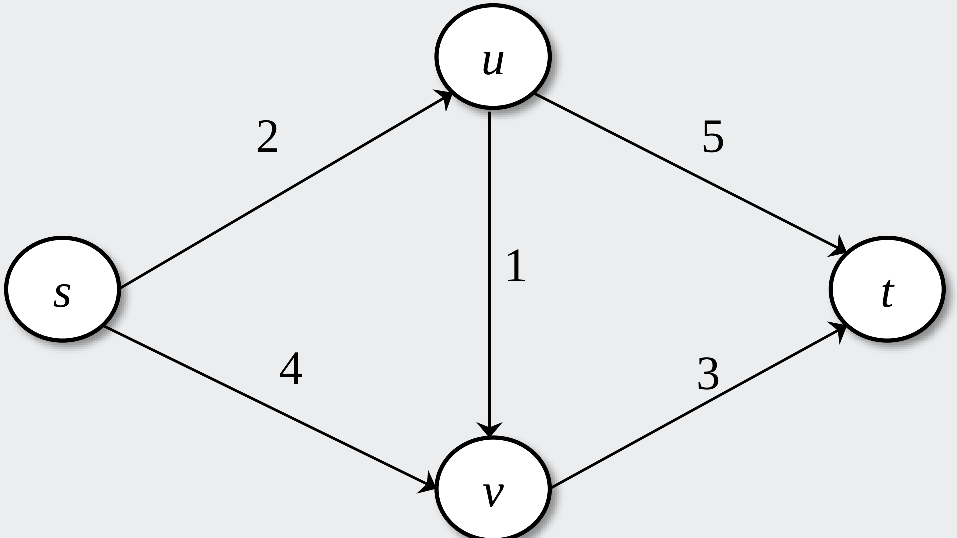
$$r_j = \begin{cases} \dfrac{d_j}{I_j}; d_j \geq 0 \\ \dfrac{-d_j}{D_j}; d_j \leq 0 \end{cases} \quad ; d_j = \text{ change in } c_j$$

100% rule: $\sum r_j \leq 1 \Rightarrow$ basis does not change. Sufficient condition, but not necessary!

- Similar rule applies to multiple coefficient changes in $\underline{b}$

Economic interpretation of dual variables

- Economic interpretation of Lagrange multipliers
  - Consider the shortest path problem again



  - *s*, *u*, *v*, *t* are computers, edge lengths are costs of sending a message between them
  - Q: What is the cheapest way to send a message from *s* to *t*?
  - Want to minimize message cost…AT&T
  - Intuitively, $x_{sv}=x_{ut}=0$ (i.e., no messages are sent from *s* to *v* and from *u* to *t*)
  - Shortest path $s \rightarrow u \rightarrow v \rightarrow t \Rightarrow x_{su}=x_{uv}=x_{vt}=1$
  - Shortest path length =2+1+3=6

LP formulation of shortest path problem

- Let x_{sv} be the fraction of messages sent from s to v
- Problem Formulation

$$\begin{aligned}
&\min 2x_{su} + 4x_{sv} + x_{uv} + 5x_{ut} + 3x_{vt} \\
&s.t. \quad x_{su}, x_{sv}, x_{uv}, x_{ut}, x_{vt} \geq 0 \\
&\qquad x_{su} - x_{uv} - x_{ut} = 0 \text{ (message not lost at } u) \\
&\qquad x_{sv} + x_{uv} - x_{vt} = 0 \text{ (message not lost at } v) \\
&\qquad x_{ut} + x_{vt} = 1 \text{ (message received at } t)
\end{aligned}$$

- Add all constraints ⇒ $x_{su} + x_{sv} = 1$ which it must be!!

  ⇒ only 3 independent constraints (although 4 nodes)
- In matrix notation:

$$A\underline{x} = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{su} \\ x_{sv} \\ x_{uv} \\ x_{ut} \\ x_{vt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{b}$$

- In general, n nodes n-1 independent equations

Dual of shortest path problem

- Dual of shortest path
  - Can view it as competition to AT&T (say, Sprint)
  - Sprint doesn't say how it gets the message from source to destination
  - Sprint announces the price of a message at each node: λ_s, λ_u, λ_v and λ_t
  - Sprint will buy at these prices at any node and sell it back at other nodes
    - ❖ λ_s = price of a message at node s (buying or selling)
    - ❖ λ_t = price of a message at node t (buying or selling)
    - ❖ Profit: $\lambda_t - \lambda_s$ price difference
    - ❖ Assume λ_s =0, since we are interested in price difference
  - To stay competitive, Sprint cannot charge more than AT&T:

$$\Rightarrow \lambda_u - \lambda_s = \lambda_u \le 2$$
$$\lambda_v \le 4$$
$$\lambda_v - \lambda_u \le 1$$
$$\lambda_t - \lambda_u \le 5$$
$$\lambda_t - \lambda_v \le 3$$

  - Sprint problem

Sprint Problem

$$\max \lambda_t = \max \underline{\lambda}^T \underline{b}$$

$$s.t. \begin{bmatrix} \lambda_u & \lambda_v & \lambda_t \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \le \begin{bmatrix} 2 & 4 & 1 & 5 & 3 \end{bmatrix}$$

$$\Rightarrow \underline{\lambda}^T A \le \underline{c}^T$$

  - Sprint maximizes its income and AT&T minimizes its cost!!
  - Lowest cost on AT&T = highest income of Sprint!!

CS condition in shortest path problem

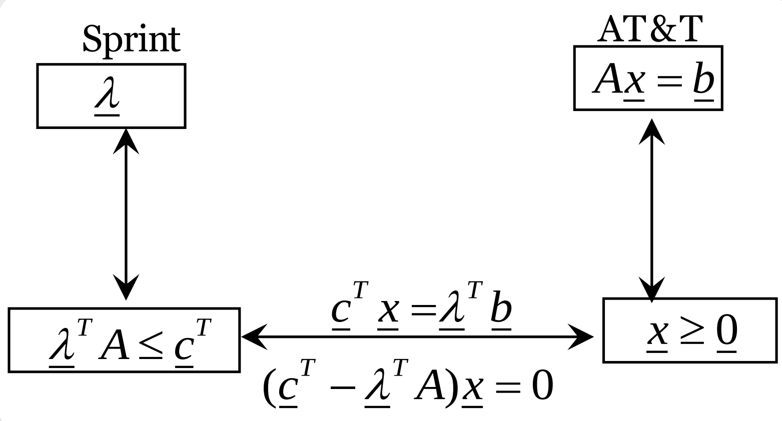
- Let us formalize these notions with our example
  - Optimal path $s \to u \to v \to t$

$$\text{Basis } B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}; \underline{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; B\underline{x}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{b}$$

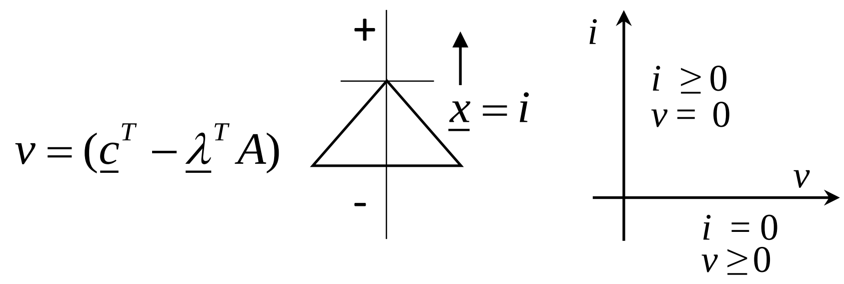
$$\underline{\lambda}^T = [2 \quad 1 \quad 3] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = [2 \quad 3 \quad 6]$$

  - Sprint prices $\lambda_u = 2$, $\lambda_v = 3$ and $\lambda_t = 6$; profit: $\lambda_t - \lambda_s = 6$
  - AT&T path: $s \to u \to v \to t$; cost: 6
- Duality: minimum cost on AT&T=maximum profit on Sprint
- Optimality:
  - $(\underline{c}^T - \underline{\lambda}^T A)\underline{x} = 0$
  - Edges in the shortest path > 0
  - On these edges, $\lambda_u - \lambda_s = \lambda_u = 2 = c_{su}$; $\lambda_v - \lambda_u = 1 = c_{uv}$; $\lambda_t - \lambda_v = 3 = c_{vt}$
  - Satisfies complementary slackness condition. Note that λ_u , λ_v , λ_t are the lengths of the shortest paths from s to the nodes u, v, and t, respectively
  - Dual can be solved by successively relaxing the dual constraints & finding the shortest paths from source to each node recursively…DIJKSTRA's algorithm

CS condition and ideal diode



Like ideal diode



- Synthetic price interpretation …inequality constrained case

Primal

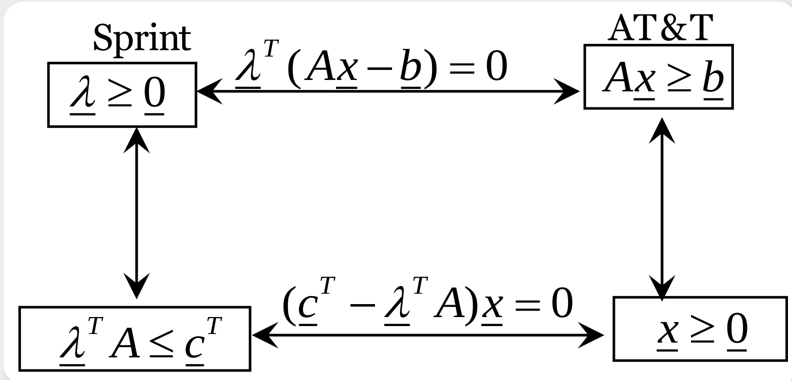
$$\min \underline{c}^T \underline{x}$$
$$s.t.\ \underline{x} \ge \underline{0}$$
$$A\underline{x} \ge \underline{b}$$

Dual

$$\max \underline{\lambda}^T \underline{b}$$
$$s.t.\ \underline{\lambda}^T A \le \underline{c}^T$$
$$\underline{\lambda} \ge \underline{0}$$

- Optimality:
  1. $(\underline{c}^T - \underline{\lambda}^T A)\underline{x} = 0$
  2. $\underline{\lambda}^T (A\underline{x} - \underline{b}) = 0$

Minimax theorem



- Top relation equivalent to grounding a node to recover KCL ⇒ conservation of current
  - If $\lambda_j > 0$ the node is above ground and KCL applies
  - If $\lambda_j = 0$ node is grounded to draw excess current $(A\underline{x}-\underline{b})_j$

- Saddle point interpretation and minimax theorem
  - Consider standard LP

$$\begin{aligned} &\min \underline{c}^T \underline{x} \\ &s.t.\ A\underline{x} = \underline{b} \\ &\quad \underline{x} \geq \underline{0} \end{aligned}$$

  - This is equivalent to:

$$\begin{aligned} &\min \underline{c}^T \underline{x} \\ &s.t.\ A\underline{x} = \underline{b} \\ &\quad \underline{x} \geq \underline{0} \end{aligned} \quad = \quad \begin{aligned} &\min_{\underline{x}} \max_{\underline{\lambda}} \left[\underline{c}^T \underline{x} + \underline{\lambda}^T (\underline{b} - A\underline{x}) \right] \\ &s.t.\ \underline{\lambda} \text{ unrestricted} \\ &\quad \underline{x} \geq \underline{0} \end{aligned}$$

  - $\underline{\lambda}$ ~ vector of Lagrange multipliers enforcing the constraint
  - If $A\underline{x} \neq \underline{b},\ |\underline{\lambda}| \to \infty$

Duality and Game Theory

- Suppose we can interchange $\underline{x}$ and $\underline{\lambda}$

$$\begin{aligned}&\max_{\underline{\lambda}}\min_{\underline{x}}\left[\left(\underline{c}^T-\underline{\lambda}^T A\right)\underline{x}+\underline{\lambda}^T\underline{b}\right]\\ &s.t.\quad \underline{\lambda}\ \text{unrestricted}\\ &\qquad\ \underline{x}\geq\underline{0}\end{aligned} \quad = \quad \begin{aligned}&\max\ \underline{\lambda}^T\underline{b}\\ &s.t.\ \underline{\lambda}\ \text{unrestricted}\\ &\quad\left(\underline{c}^T-\underline{\lambda}^T A\right)\geq\underline{0}\end{aligned}$$

- Note: Don't get minimum = -∞ if $(\underline{c}^T-\underline{\lambda}^T A)\geq\underline{0}\Rightarrow\underline{x}=\underline{0}$
- So, duality is equivalent to finding the saddle point $(\underline{x}^*,\underline{\lambda}^*)$ that maximizes $L(\underline{x},\underline{\lambda})=\underline{c}^T\underline{x}-\underline{\lambda}^T A\underline{x}+\underline{\lambda}^T\underline{b}$ w.r.t $\underline{\lambda}$ and that minimizes $L(\underline{x},\underline{\lambda})$ w.r.t $\underline{x}$

$$\begin{aligned}&\min_{\underline{x}}\max_{\underline{\lambda}} L(\underline{x},\underline{\lambda})\\ &s.t.\ \underline{\lambda}\ \text{unrestricted}\\ &\quad\ \underline{x}\geq\underline{0}\end{aligned} \quad = \quad \begin{aligned}&\max_{\underline{\lambda}}\min_{\underline{x}} L(\underline{x},\underline{\lambda})\\ &s.t.\ \underline{\lambda}\ \text{unrestricted}\\ &\quad\ \underline{x}\geq\underline{0}\end{aligned}$$

- This is called minimax theorem

- Game Theory: Suppose we have two decision makers (players) y and z
  - y is the row player; y chooses one of m strategies
  - z is the column player; z chooses one of n strategies
  - If the row player chooses strategy i and column player chooses strategy j, the row player *receives* a reward of a_{ij} and the column player *loses* an amount a_{ij}
  - Such a game is called **a two person zero-sum game**

Minimax Strategies

- Example

Row Strategy	Column	Player	Strategy	Row Min.
	Column 1	Column 2	Column 3	
Row 1	4	4	10	4
Row 2	2	3	1	1
Row 3	6	5	7	5
Col. Max.	6	5	10	

Saddle point condition:

$$\max_{\text{all rows}} (\text{row minimum}) = \min_{\text{all columns}} (\text{column maximum})$$

Neither player can unilaterally change strategy and benefit.

Q: Are all strategies pure? NO!

- **Mixed (Randomized) Strategy**: Suppose we have two football coaches y and z
  - z is the offensive (column) coach and y is the defensive (row) coach
  - z chooses between run and pass
  - y chooses defense against run or pass
  - To fix ideas, suppose if y defends against a run and z chooses to run he gains 1 yard. On the other hand if z chooses to pass, he gets 7 yards
  - If y defends against a pass and z chooses to run, he gets 5 yards. On the other hand, if z chooses to pass, he loses 5 yards

Minimax Randomized strategies

$$\overbrace{\begin{matrix}\text{Run} & \text{Pass}\end{matrix}}^{z}$$

$$A = \begin{bmatrix} 1 & 7 \\ 5 & -5 \end{bmatrix} \begin{matrix} \text{Defend against run} \\ \text{Defend against pass} \end{matrix} \Bigg\} \quad y$$

- Pay-off matrix for y = -pay-off matrix for z
- y and z must employ mixed randomized strategies
- If z always runs, he cannot make it (the opponent can learn and defend against run!)
- Suppose λ_1 is the probability that z will run, (1- λ_1) is the probability of pass
- Expected gain $\lambda_1 + 7 - 7\lambda_1 = 7 - 6\lambda_1$ if y defends against run

  $5\lambda_1 - 5 + 5\lambda_1 = 10\lambda_1 - 5$ if y defends against pass
- y would minimize z's gain. z will maximize the minimum gain
- Note: $7 - 6\lambda_1$ decreases with λ_1 , while $10\lambda_1 - 5$ increases
- Optimum when $7 - 6\lambda_1 = 10\lambda_1 - 5 \Rightarrow \lambda_1 = 12/16 = 3/4$

  $\Rightarrow$ offense should run ¾ of the time

  $\Rightarrow$ expected gain: 7- (18/4) = 2.5 yards
- What about y?
  - y will minimize the maximum
  - Expected gain of z $\quad x_1 + 5 - 5x_1 = 5 - 4x_1$ if z chooses to run

    $7x_1 - 5 + 5x_1 = 12x_1 - 5$ if z chooses to pass

Minimax Theorem and Duality

- Minimize maximum gain ⇒ $x_1 = 5/8$
- Expected gain of z: 7 – (18/4) = 2.5 yards
- Neither player can do better by making a change

- A simple derivation of minimax theorem of game theory
  - Two players y and z

$$\underbrace{\overbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & \cdots \\ \vdots & & & \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}^{z}}_{\text{rows: } y}$$

  - Pay-off to $f = \underline{y}^T A \underline{z}$
  - Consider the minimax problem

$$\begin{aligned} &\min_{\underline{y}} \max_{\underline{z}} \underline{y}^T A \underline{z} \\ s.t.\quad & \underline{z}^T \underline{e} = 1 \\ & \underline{y}^T \underline{e} = 1 \\ & \underline{z} \geq \underline{0} \\ & \underline{y} \geq \underline{0} \end{aligned} \quad \underset{\text{dualize } \underline{z} \text{ constraint}}{\Rightarrow} \quad \begin{aligned} &\min_{\underline{y},\alpha} \alpha \\ s.t.\quad & A^T \underline{y} \leq \alpha \underline{e} \\ & \underline{y}^T \underline{e} = 1 \\ & \underline{y} \geq \underline{0} \end{aligned}$$

  - Recall

$$\begin{aligned} &\max_{\underline{z}} \underline{y}^T A \underline{z} \\ s.t.\quad & \underline{z}^T \underline{e} = 1 \\ & \underline{z} \geq \underline{0} \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\min \alpha \\ s.t.\quad & \underline{e}\alpha \geq A^T \underline{y} \end{aligned}$$

Proof of Minimax theorem

- Alternatively, consider maximin problem

$$\max_{\underline{z}} \min_{\underline{y}} \underline{y}^T A \underline{z} \quad s.t. \quad \underline{e}^T \underline{z} = 1,\ \underline{e}^T \underline{y} = 1,\ \underline{z} \geq \underline{0},\ \underline{y} \geq \underline{0}$$

$\Rightarrow$ dualize $\underline{y}$ constraint

$$\max_{\underline{z},\mu} \mu \quad s.t. \quad A\underline{z} \leq \mu \underline{e},\ \underline{e}^T \underline{z} = 1,\ \underline{z} \geq \underline{0}$$

- To prove equality
  - Let

$$\underline{x} = \left(\frac{1}{\mu}\right)\underline{z};\ \underline{\lambda} = \left(\frac{\underline{y}}{\alpha}\right)$$

$$\underline{x}^T \underline{e} = \left(\frac{1}{\mu}\right)\underline{z}^T \underline{e}$$

$$\Rightarrow \underline{x}^T \underline{e} = \frac{1}{\mu} \ \&\ \underline{\lambda}^T \underline{e} = \frac{1}{\alpha}$$

maximin

$$\min \underline{x}^T \underline{e} \quad s.t. \quad A\underline{x} \geq \underline{e},\ \underline{x} \geq \underline{0}$$

$\Rightarrow$

minimax

$$\max \underline{\lambda}^T \underline{e} \quad s.t. \quad A^T \underline{\lambda} \leq \underline{e},\ \underline{\lambda} \geq \underline{0}$$

- From duality theorem maximin ≡ minimax
- **You can always add a constant to all elements of *A* so that μ and α > 0.**

Stone, Paper, Scissors Problem

- Reward structure for row player: *Stone* $\succ$ *Scissors*; *Scissors* $\succ$ *Paper*; *Paper* $\succ$ *Stone*

Row Strategy	Column	Player	Strategy	Row Min.
	Stone	Paper	Scissors	
Stone	0	-1	1	-1
Paper	1	0	-1	-1
Scissors	-1	1	0	-1
Col. Max.	1	1	1	

y game:Maximin

$$\max \lambda_1 + \lambda_2 + \lambda_3$$
$$s.t. \quad \lambda_1 + 2\lambda_2 \le 1$$
$$\lambda_2 + 2\lambda_3 \le 1 \Rightarrow \lambda_i = \frac{1}{3}; \alpha = 1$$
$$\lambda_3 + 2\lambda_1 \le 1$$
$$\underline{\lambda} \ge \underline{0}$$

Add 1 to each element of matrix *A*

Row Strategy	Column	Player	Strategy	Row Min.
	Stone	Paper	Scissors	
Stone	1	0	2	0
Paper	2	1	0	0
Scissors	0	2	1	0
Col. Max.	2	2	2	

z game:Minimax

$$\min x_1 + x_2 + x_3$$
$$s.t. \quad x_1 + 2x_3 \ge 1$$
$$2x_1 + x_2 \ge 1 \Rightarrow x_i = \frac{1}{3}; \mu = 1$$
$$2x_2 + x_3 \ge 1$$
$$\underline{x} \ge \underline{0}$$

Reward of original game = 0

Other interesting game problems

- Two person non-constant sum games: Example: Prisoner's dilemma

Prisoner 1	Prisoner 2	
	Confess	Don't confess
Confess	(-5,-5)	(0,-20)
Don't confess	(-20,0)	(-1,-1)

Equilibrium strategy: (-5, -5)

- Non-cooperative Game Theory
  - Nash equilibrium, Bayesian games,…
- Cooperative game theory with N decision makers
  - Now, you can form coalitions
  - Characteristic function of a coalition $v(S)$, $S \subseteq N=\{1,2,3,..,N\}$
  - Core of a game: Undominated reward imputations
  - Sahpley value: How should rewards be allocated equitably?
- Incentives, Auctions and Mechanism Design

Finding the core is equivalent to solving a system of linear inequalities

Good book: Y. Narahari, *Game Theory and Mechanism Design*, World Scientific,2014.

Duality and Decomposition

- Strategies for solving large-scale mathematical programming problems
  - Separable Problems

$$\min_{\underline{x}_1,\ldots,\underline{x}_r} \sum_{i=1}^{r} f_i(\underline{x}_i)$$
$$s.t. \quad \underline{x}_i \in \Omega_i; i = 1,\ldots,r$$

❖ Due to separability, can solve *r* decoupled problems

$$\begin{aligned}
&\text{for } i = 1,\ldots,r \\
&\qquad \min_{\underline{x}_i} f_i(\underline{x}_i) \\
&\qquad s.t. \;\; \underline{x}_i \in \Omega_i \\
&\text{end}
\end{aligned}$$

  - Dantzig-Wolf decomposition . . . price-directed decomposition

$$\begin{aligned}
&\min \underline{c}^T \underline{x} \\
&s.t. \quad A\underline{x} \ge \underline{b} \\
&\qquad \bar{A}\underline{x} \ge \underline{b} \\
&\qquad \underline{x} \ge \underline{0}
\end{aligned}$$

❖ To illustrate the method consider

Let $\underline{X} = \{\underline{x} : \underline{x} \ge \underline{0}, A\underline{x} \ge \underline{b}\}$

Further, let $\{\underline{x}_1,\ldots,\underline{x}_p\}$ be the extreme points of this set. Then:

$$\min_{x \in \underline{X}} \underline{c}^T \underline{x} \quad s.t. \quad \bar{A}\underline{x} \ge \bar{\underline{b}}$$

Application of Duality

- This LP can be rewritten using

  Let $\underline{x} = \sum_{j=1}^{p} \alpha_j x_j; \; \sum_{j=1}^{p} \alpha_j = 1$

  then the above LP is equivalent to:

$$\min_{\underline{\alpha} \geq \underline{0}} \underline{c}^T \left(\sum_{j=1}^{p} \alpha_j x_j \right)$$

$$s.t. \qquad \sum_{j=1}^{p} \alpha_j = 1$$

$$\bar{A} \left(\sum_{j=1}^{p} \alpha_j x_j \right) \geq \underline{\bar{b}}$$

- At optimum, we need $\underline{\lambda} \geq \underline{0}$ and

$$\underline{c}^T \underline{x}_j - \lambda_0 - \underline{\lambda}^T \bar{A} \underline{x}_j \geq \underline{0}; \; j = 1, \ldots, p$$

- So need

$$\min_{1 \leq j \leq p} \left(\underline{c}^T - \underline{\lambda}^T \bar{A} \right) \underline{x}_j - \lambda_0 \geq \underline{0}$$

- or

$$\min_{\underline{x} \in \bar{\underline{x}}} \left(\underline{c}^T - \underline{\lambda}^T \bar{A} \right) \underline{x} - \lambda_0 \geq \underline{0}$$

Application of Duality

- Note that if

$$\overline{A} = \begin{bmatrix} A_1 & \cdots & \cdots & \vdots \\ \vdots & A_2 & \vdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \cdots & A_r \end{bmatrix}$$

Recall that this is related to Column generation method

- The minimization problem decouples into r sub-problems
- Coordinator sets the prices and subordinates solve subproblems using specified prices

○ Activity-directed decomposition.......Bender's method

$$\begin{aligned} &\min_{\underline{x}\geq\underline{0},\underline{y}\in Y} \underline{c}^T \underline{x} + f(\underline{y}) \\ &s.t. \qquad A\underline{x} + F(\underline{y}) \geq \underline{b} \end{aligned}$$

- The minimization can be written as a nested minimization (also called projection)

$$\min_{\underline{y}\in Y} \left[f(\underline{y}) + \min_{\underline{x}\geq\underline{0}} \left\{ \underline{c}^T \underline{x} \quad s.t. \quad A\underline{x} \geq \underline{b} - F(\underline{y}) \right\} \right]$$

- So we need to solve the LP: $\min_{\underline{x}\geq\underline{0}} \underline{c}^T \underline{x}$

$$s.t. \qquad A\underline{x} \geq \underline{b} - F(\underline{y})$$

- The dual is $\max_{\underline{\lambda}\geq\underline{0}} \underline{\lambda}^T \left(\underline{b} - F(\underline{y})\right)$

$$s.t. \qquad \underline{\lambda}^T A \leq \underline{c}^T$$

Application of Duality

- So the original problem is equivalent to

$$\min_{\underline{y}\in Y}\left[f\left(\underline{y}\right)+\max_{\underline{\lambda}\geq \underline{0}}\left\{ \underline{\lambda}^T\left(\underline{b}-F\left(\underline{y}\right)\right)\ \ s.t.\ \underline{\lambda}^T A\leq \underline{c}^T \right\}\right]$$

- Since

$$\max_{\underline{\lambda}\geq \underline{0}}\left\{ \underline{\lambda}^T\left(\underline{b}-F\left(\underline{y}\right)\right)\ \ s.t.\ \underline{\lambda}^T A\leq \underline{c}^T \right\}=\max_{1\leq j\leq p}\underline{\lambda}_j^T\left(\underline{b}-F\left(\underline{y}\right)\right)$$

- Where $\{\underline{\lambda}_j\}$ are the extreme points of the set:

$$\left\{ \underline{\lambda} : \underline{\lambda}\geq \underline{0} \text{ and } \underline{\lambda}^T A\leq \underline{c}^T \right\}$$

$$\min_{\underline{y}\in Y}\left[f\left(\underline{y}\right)+y_0\right]$$

$$\Rightarrow s.t.\quad y_0\geq \underline{\lambda}_j^T\left(\underline{b}-F\left(\underline{y}\right)\right)$$

- Algorithm procedure
    - Start with a trial $(\hat{\underline{y}}, \hat{y}_0)$
    - Solve the LP to get $\underline{\lambda}$ (and $\underline{x}$ = multipliers)… optimum value of z^*
    - If $\hat{y}_0 \geq z^* \Rightarrow$ done
    - Else set $\hat{y}_0 = z^*$, optimize over $\underline{y}$ to get new $\hat{\underline{y}}$
- Need convexity of $f(y)$ and the feasible set of Y for convergence
- The above procedure goes under the name of Bender's decomposition or activity – directed decomposition

- Resource-directed decomposition
    - Consider the same problem as in Dantzig-Wolf decomposition

Application of Duality

$$\min_{\underline{x}_1,\ldots,\underline{x}_r} \sum_{i=1}^{r} \underline{c}_i^T \underline{x}_i$$

$$\Rightarrow s.t. \quad A_i \underline{x}_i \geq \underline{b}_i$$

$$\sum_{i=1}^{r} B_i \underline{x}_i \geq \underline{b}_0$$

- Split resource vector $\underline{b}_0$ into r parts $\underline{y}_1,\ldots,\underline{y}_r$ then

$$\min_{\underline{y}_1,\ldots,\underline{y}_r} \sum_{i=1}^{r} v_i(\underline{y}_i) \qquad s.t. \quad \sum_{i=1}^{r} \underline{y}_i \geq \underline{b}_0$$

where

$$v_i\left(\underline{y}_i\right) = \min_{\underline{x}_i} \underline{c}_i^T \left(\underline{x}_i\right) \qquad s.t. \quad A_i \underline{x}_i \geq \underline{b}_i, \quad B_i \underline{x}_i \geq \underline{y}_i$$

- Updating $\underline{y}_i$ is a little more complex here
- Need to find a feasible direction that guarantees a decrease in cost or use subgradient method

■ Non-linear version of decomposition methods… ECE 6437

- Consider

$$\min_{\underline{x}} \sum_{i=1}^{r} f_i\left(\underline{x}_i\right)$$

$$s.t. \quad \sum_{i=1}^{r} g_i\left(\underline{x}_i\right) \leq \underline{b}$$

- The problem can be viewed as a two-level scheme
  - ❖ Coordinator-level: Maximize with respect to $\underline{\lambda}$
  - ❖ Subordinate level: solve r sub-problems

$$\max_{\underline{\lambda} \geq 0} \left[\underline{\lambda}^T \underline{b} + \sum_{i=1}^{r} \min_{\underline{x}_i} \left\{ f_i\left(\underline{x}_i\right) - \underline{\lambda}^T g_i\left(\underline{x}_i\right) \right\} \right]$$

Duality Summary

- Summary
  - Duality
    - SLP ⇒ asymmetric dual
    - Inequality constraints ⇒ symmetric dual
    - Unconstrained variable ⇒ equality constraint in dual
  - Properties
    - Minimum of primal ≡ maximum of dual
    - Dual of dual ≡ primal
    - Interpretations as shadow prices
    - Useful in sensitivity analysis (see chapter 5 of Bertsimas and Tsitsiklis)
  - Applications of duality to solve large-scale mathematical programming problems .....more to come from lecture 6 onwards