Lecture 5:
Dual Simplex, Primal – Dual And Karmarkar's Algorithms

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Outline

- Review of duality
- Dual simplex algorithm
  - Revised simplex: primal feasibility $\xrightarrow{\text{work towards}}$ dual feasibility
  - Dual simplex: dual feasibility $\xrightarrow{\text{work towards}}$ primal feasibility
- Primal-dual algorithm
  - Enforce complementary slackness conditions over subsets of $\{1,2, \dots, n\}$
  - Widely used to solve network flow, assignment & transportation problems
- Interior point methods
  - The primal path following algorithm
  - Affine scaling methods (see notes. Will not be covered)
  - The potential reduction algorithm
  - The primal-dual path following algorithm
  - Implementation issues
- Comparison of revised simplex and Interior point methods
- Summary

Review of uality

- Duality
  - SLP and its dual

$$\begin{aligned} &\min \underline{c}^T \underline{x} \\ &s.t.\ A\underline{x} = \underline{b} \\ &\underline{x} \geq \underline{0} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\max \underline{\lambda}^T \underline{b} \\ &s.t.\ \underline{\lambda} \text{ unrestricted} \\ &\underline{\lambda}^T A \leq \underline{c}^T \end{aligned}$$

    - Asymmetric form of the dual
  - Inequality constrained LP and its dual

$$\begin{aligned} &\min \underline{c}^T \underline{x} \\ &s.t.\ A\underline{x} \geq \underline{b} \\ &\underline{x} \geq \underline{0} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\max \underline{\lambda}^T \underline{b} \\ &s.t.\ \underline{\lambda} \geq \underline{0} \\ &\underline{\lambda}^T A \leq \underline{c}^T \end{aligned}$$

    - Symmetric form of the dual
  - For all feasible $\underline{x}$ in primal and $\underline{\lambda}$ in dual
    - $\underline{\lambda}^T \underline{b} \leq \underline{c}^T \underline{x}$ $\Rightarrow$ dual feasible solution is always a lower bound on the primal
    - Dual unbounded $\Rightarrow$ primal infeasibility
    - Primal unbounded $\Rightarrow$ dual infeasibility
    - Primal infeasibility may imply dual infeasibility and vice-versa
    - When dual and primal have finite optimal solution, max of the dual, $\underline{\lambda}^T \underline{b}$ = min of the primal, $\underline{c}^T \underline{x}^*$

Complementary Slackness & Sensitivity

- Complementary slackness conditions

$$(\underline{c}^T - \underline{\lambda}^{*T} A)\underline{x}^* = 0 \Rightarrow x_i^* > 0 \Rightarrow c_i = \underline{\lambda}^T \underline{a}_i$$

(or relative cost = 0 or x_i^* in basis)

$$\Rightarrow x_i^* = 0 \Rightarrow c_i > \underline{\lambda}^T \underline{a}_i$$

(or relative cost > 0 or x_i^* is nonbasic)

$\Rightarrow$ **true cost > synthetic cost**

- For inequality constrained problem

$$(\underline{\lambda}^*)^T (A\underline{x} - \underline{b}) = 0$$

$$\Rightarrow \lambda_i^* > 0 \Rightarrow \quad \underline{a}_i^T \underline{x} = b_i \quad \text{(nonbasic surplus)}$$

$$\lambda_i^* = 0 \Rightarrow \quad \underline{a}_i^T \underline{x} > b_i \quad \text{(basic surplus)}$$

- Simplex multipliers λ_j are the costs of $\underline{e}_j$, the jth unit vector
- Cost of any other vector $\underline{a}_k$ is $\sum \lambda_j a_{jk} = \underline{\lambda}^T \underline{a}_{k\ldots}$ synthetic cost of vector $\underline{a}_k$
  - $\lambda_j^* = \dfrac{\partial f}{\partial b_j}; \; x_j^* = \dfrac{\partial f}{\partial c_j}$

Dual Simplex Algorithm

- In the shortest path problem, λ_j can be interpreted as the length of the shortest path from source to node j
  - If $\lambda_j - \lambda_i = c_{ij}$, edge (i, j) is in the shortest path
  - If $\lambda_j - \lambda_i < c_{ij}$, edge (i, j) is not in the shortest path
  - $\underline{\lambda}^*$ and $\underline{x}^*$ are saddle points of

$$L(\underline{x}, \underline{\lambda}) = \underline{c}^T \underline{x} - \underline{\lambda}^T A\underline{x} + \underline{\lambda}^T \underline{b}$$

$$\Rightarrow \min_{\underline{x}\geq 0} \max_{\underline{\lambda}} L(\underline{x}, \underline{\lambda}) = \max_{\underline{\lambda}} \min_{\underline{x}\geq 0} L(\underline{x}, \underline{\lambda})$$

- **Dual Simplex Algorithm**
  - Primal revised simplex starts with a primal feasible solution $\underline{x}$ s.t. $A\underline{x} = \underline{b}, \underline{x} > \underline{0}$ and work towards $(\underline{c}^T - \underline{\lambda}^T A) = \underline{p}^T \geq \underline{0} \Rightarrow$ dual feasibility

$$A\underline{x} = \underline{b}, \quad \underline{x} \geq \underline{0} \quad \xrightarrow{\text{update } \underline{x}} \quad \underline{c}^T - \underline{\lambda}^T A \geq \underline{0}, \quad \underline{\lambda} = \underline{c}_B^T B^{-1}$$

  - Note
    - Basic $\Rightarrow$ equality
    - Non-basic $\Rightarrow$ strict inequality

From Dual Feasibility to Primal Feasibility

- What if we tried another approach?

$$\begin{array}{ccc} \text{From Dual Feasibility} & \rightarrow & \text{Primal Feasibility} \\ \underline{c}^T - \underline{\lambda}^T A \geq \underline{0} & \xrightarrow{\text{update } \lambda} & \underline{x}_B = B^{-1}\underline{b},\ \underline{x}_B \geq \underline{0} \end{array}$$

- The latter approach leads to the *Dual Simplex Algorithm*
- Key ideas:
  - Suppose $\underline{\lambda}$ is dual feasible

    $$\Rightarrow \underline{\lambda}^T A \leq \underline{c}^T \text{ or } \underline{\lambda}^T \underline{a}_j \leq c_j \forall j$$

  - Suppose our basis B consists of the first m columns

    $$(\underline{a}_1,\ \underline{a}_2,\ \ldots,\ \underline{a}_m)$$

  - From revised simplex and complementary slackness conditions, we know

    $$\underline{\lambda}^T \underline{a}_j = c_j;\ 1 \leq j \leq m \quad \Rightarrow \underline{\lambda}^T = \underline{c}_B^T B^{-1}$$

    $$\underline{\lambda}^T \underline{a}_j < c_j;\ m+1 \leq j \leq n \quad \text{(barring degeneracy)}$$

- What is the corresponding $\underline{x}_B = B^{-1}\underline{b}$ (is it primal feasible?)

Need not be Primal Feasible!!

- Suppose $x_{Bl} < 0$, we must remove the corresponding column $\underline{a}_l$ from the basis
  - $x_{Bl} = \left[\text{row } l \text{ of } \left(B^{-1}\right)\right] * \underline{b}$

Dual Step Size Selection

- Since want to maximize the dual, what if I perturb $\underline{\lambda} \to \underline{\hat{\lambda}}$ s. t.

  $$\underline{\hat{\lambda}}^T \underline{b} = \underline{\lambda}^T \underline{b} - \varepsilon x_{Bl} > \underline{\lambda}^T \underline{b},\ \varepsilon > 0, = (\underline{\lambda}^T - \varepsilon \text{ row } l(B^{-1}))\underline{b}$$

- So, $\underline{\hat{\lambda}}^T = \underline{\lambda}^T - \varepsilon \text{ row } l(B^{-1}) = (\underline{c}_B^T - \varepsilon \underline{e}_l^T)B^{-1}$

- *Q: How far to go?*
- *A: Only so far as to maintain dual feasibility*

$$(\underline{c}^T - \underline{\hat{\lambda}}^T A) \geq \underline{0}^T$$

$$\underline{\hat{\lambda}}^T \underline{a}_j = c_j,\ j \neq l,\ j = 1, \ldots, m$$

$$\underline{\hat{\lambda}}^T \underline{a}_l = c_l - \varepsilon < c_l \quad \text{(out of the basis)}$$

$$\underline{\hat{\lambda}}^T \underline{a}_j = \underline{\lambda}^T \underline{a}_j - \varepsilon \underline{e}_l^T B^{-1} \underline{a}_j,\quad j = m+1, \ldots, n$$

$$= z_j - \varepsilon \alpha_{lj},\quad j = m+1, \ldots, n \quad \text{where } z_j < c_j$$

- *What does this mean:* $\underline{\hat{\lambda}}^T \underline{a}_l < c_l \Rightarrow$ strict inequality or column $\underline{a}_l$ left the basis
- *Q: Which column should we bring into the basis?*
- *A: The one that makes* $z_j - \varepsilon \alpha_{lj} = c_j$ *first*
- What if all $\alpha_{lj} \geq 0$?

  $\Rightarrow$ Can never make $c_j = z_j - \varepsilon \alpha_{lj}$ since $z_j < c_j$

  $\Rightarrow$ Dual unbounded, since $\underline{\hat{\lambda}}$ is feasible $\forall \varepsilon$

Dual Simplex Algorithm Steps

- If any $\alpha_{lj} < 0$, can move until $\varepsilon_j = \frac{z_j - c_j}{\alpha_{lj}} = \frac{-p_j}{\alpha_{lj}}$

$\Rightarrow$Among these ε, pick one that reaches c_j first $\varepsilon = \frac{z_k - c_k}{\alpha_{lk}} = \frac{-p_k}{\alpha_{lk}} = \min_j \left\{ \frac{z_j - c_j}{\alpha_{lj}} : \alpha_{lj} < 0 \right\}$

- Update basis $B = B - \text{column } \underline{a}_l + \text{column } \underline{a}_k$ as in revised simplex and compute $\underline{x}_B = B^{-1}\underline{b}$

- **Dual simplex algorithm steps**:

<u>Step 1</u>: Given a dual feasible solution $\underline{x}_B = B^{-1}\underline{b}$

if $\underline{x}_B \geq \underline{0}$ then the solution is optimal

else select an index l such that $x_{Bl} < 0$

<u>Step 2</u>: If all $\alpha_{lj} = \left[\text{row } l \text{ of } (B^{-1})\right] * \underline{a}_j \geq 0$ for all non-basic columns $\underline{a}_j$,

then unbounded dual (or infeasible primal)

else $\varepsilon = \min_j \left\{ \frac{z_j - c_j}{\alpha_{lj}} = \frac{-p_j}{\alpha_{lj}} : \alpha_{lj} < 0 \right\} = \frac{z_k - c_k}{\alpha_{lk}} = \frac{-p_k}{\alpha_{lk}}$

<u>Step 3</u>: Update $\underline{\lambda}$, basis B, and $\underline{x}_B$

$$\underline{\lambda}^T \leftarrow \underline{\lambda}^T - \varepsilon \text{ row } l(B^{-1})$$

$$B \leftarrow B - \text{column } \underline{a}_l + \text{column } \underline{a}_k \text{ (or propogate } B^{-1} \text{ or } LU \text{ or } QR \text{ factors)}$$

Go back to Step 1

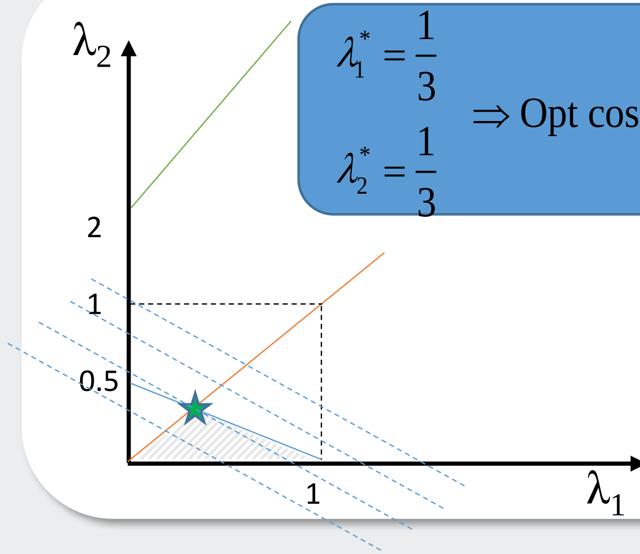
Optimality ⇒ Dual Feasibility & Primal Feasibility

- Why does it converge?
  - Maintain dual feasibility at each stage
  - Choice of $x_{Bl} < 0 \Rightarrow$ dual objective increases
  - Cannot terminate at a non-optimum point (because all we require for optimum is dual and primal feasibility)
  - Finite number of extreme points ⇒ must terminate in a finite number of steps
- Example: Primal

$$\min x_1 + 2x_2$$
$$s.t.\ x_1 - 2x_2 + x_3 \geq 4$$
$$2x_1 + x_2 - x_3 \geq 6$$
$$x_1, x_2, x_3, \geq 0$$

Dual

$$\max 4\lambda_1 + 6\lambda_2$$
$$s.t.\ \lambda_1 + 2\lambda_2 \leq 1$$
$$-2\lambda_1 + \lambda_2 \leq 2$$
$$\lambda_1 - \lambda_2 \leq 0$$
$$\lambda_1, \lambda_2 \geq 0$$



$$\lambda_1^* = \frac{1}{3} \quad \lambda_2^* = \frac{1}{3} \Rightarrow \text{Opt cost} = \frac{10}{3}$$

- Graphical Solution:

$$B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ \frac{2}{3} \end{bmatrix} \qquad x_1 = \frac{10}{3},\ x_2 = 0,\ x_3 = \frac{2}{3}$$

$$\underline{\lambda}^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix} \qquad \text{opt cost} = \frac{10}{3}$$

Illustration of Dual Simplex Algorithm

- Example:

Primal

$$\min 3x_1 + 4x_2 + 5x_3$$
$$s.t.\;\; x_1 + 2x_2 + 3x_3 \geq 5$$
$$2x_1 + 2x_2 + x_3 \geq 6$$
$$x_i \geq 0$$

Dual

$$\max 5\lambda_1 + 6\lambda_2$$
$$s.t.\;\; \lambda_1 + 2\lambda_2 \leq 3$$
$$2\lambda_1 + 2\lambda_2 \leq 4$$
$$3\lambda_1 + \lambda_2 \leq 5$$
$$\lambda_1, \lambda_2 \geq 0$$

Optimal Solution:

$$\lambda_1 = 1,\; \lambda_2 = 1 \Rightarrow x_1 = 1,\; x_2 = 2,\; x_3 = 0$$
$$\text{optimal cost} = 11$$

Iteration 0:

(1):

$$\lambda_1 = \lambda_2 = 0 \Rightarrow z_j = 0\; \forall j$$
$$x_1 + 2x_2 + 3x_3 - s_1 = 5$$
$$2x_1 + 2x_2 + x_3 - s_2 = 6$$
$$\Rightarrow B = -I \text{ is the basis}$$
$$\underline{x}_B = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}$$

Select the most negative one : s_2

(2):

$$p_1 = c_1 - z_1 = 3;\; p_2 = c_2 - z_2 = 4;\; p_3 = c_3 - z_3 = 5$$
$$(\text{row } l \text{ of } B^{-1})\underline{a}_j = -\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -1 \end{bmatrix}$$
$$\varepsilon = \min_j \left\{ \frac{z_j - c_j}{\alpha_{lj}} : \alpha_{lj} < 0 \right\} = \min \begin{bmatrix} \frac{3}{2} & \frac{4}{2} & \frac{5}{1} \end{bmatrix} = \frac{3}{2}$$

Dual Simplex Algorithm Steps

(3) Update $\underline{\lambda}$, B and $\underline{x}_B$

$$\Rightarrow \text{column 1 comes into the basis} \Rightarrow \text{basis} \begin{pmatrix} s_1 \\ x_1 \end{pmatrix}$$

$$\text{or } \underline{\lambda}^T = \underline{\lambda}^T - \varepsilon(\text{row}_l \text{ of } (B^{-1})) = \begin{bmatrix} 0 & 0 \end{bmatrix} - \tfrac{3}{2}\begin{bmatrix} 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \tfrac{3}{2} \end{bmatrix}$$

$$\text{new } B = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \text{new } B^{-1} = \begin{bmatrix} -1 & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} \end{bmatrix}$$

$$\underline{\lambda}^T = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & \tfrac{3}{2} \end{bmatrix}$$

$$\underline{x}_B = \begin{bmatrix} -1 & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \Rightarrow \begin{pmatrix} s_1 \\ x_1 \end{pmatrix}$$

<u>Iteration 1</u>:

(1) s_1 goes out of basis

(2) $$(\text{row}_1 \text{ of } B^{-1}) \underbrace{\begin{bmatrix} 2 & 3 & 0 \\ 2 & 1 & -1 \end{bmatrix}}_{N} = \begin{bmatrix} -1 & \tfrac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -\tfrac{5}{2} & -\tfrac{1}{2} \end{bmatrix}$$

Dual Simplex Algorithm

$$(\text{row}_1 \text{ of } B^{-1})N = \begin{bmatrix} -1 & -\frac{5}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\underline{\lambda}^T A - \underline{c} = \begin{bmatrix} 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 2 & 2 & 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 5 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \overset{\downarrow}{0} & -1 & -\frac{7}{2} & \overset{\downarrow}{0} & -\frac{3}{2} \end{bmatrix} = -\underline{p}$$

$$\varepsilon = \min\begin{bmatrix} \frac{1}{1} & \frac{7}{5} & 3 \end{bmatrix} \Rightarrow \text{column 2 enters the basis}$$

(3) $\underline{\lambda}^T = \underline{\lambda}^T - \varepsilon(\text{row}_1 \text{ of } B^{-1}) = \begin{bmatrix} 0 & \frac{3}{2} \end{bmatrix} - 1\begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$\text{new } B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \qquad \text{new } B^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix}$$

$$\text{Old } B^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\text{check}: \quad c_B^T B^{-1} = \begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\underline{x}_B = B^{-1}\underline{b} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$x_1 = 1, x_2 = 2, x_3 = 0$ Done!!!

Another Example of Dual Simplex Algorithm

- Example:

Primal:

x_1 = number of barrels of light crude

x_2 = number of barrels of heavy crude

$$\min\ 56x_1 + 50x_2$$

$$s.t.\ 0.3x_1 + 0.3x_2 \ge 900{,}000$$
$$0.2x_1 + 0.4x_2 \ge 800{,}000$$
$$0.3x_1 + 0.2x_2 \ge 500{,}000$$
$$x_1 \ge 0; x_2 \ge 0$$

optimal point: $(0, 3M)$; Cost: \$150M

Dual:

$$\max\ 100{,}000[9\lambda_1 + 8\lambda_2 + 5\lambda_3]$$

$$s.t.\ 0.3\lambda_1 + 0.2\lambda_2 + 0.3\lambda_3 \le 56$$
$$0.3\lambda_1 + 0.4\lambda_2 + 0.2\lambda_3 \le 50$$
$$s.t.\ \lambda_1 \ge 0; \lambda_2 \ge 0; \lambda_3 \ge 0$$

optimal point: $(500/3 \;\; 0 \;\; 0)$

Cost: \$150M

Iteration 0:

(1):

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow z_j = 0\ \forall j$$
$$0.3x_1 + 0.3x_2 - s_1 = 900{,}000$$
$$0.2x_1 + 0.4x_2 - s_2 = 800{,}000$$
$$0.3x_1 + 0.2x_2 - s_3 = 500{,}000$$
$$x_1 \ge 0; x_2 \ge 0; s_i \ge 0$$

$\Rightarrow B = -I$ is the basis

$$\underline{x}_B = -\begin{bmatrix} -900{,}000 \\ -800{,}000 \\ -500{,}000 \end{bmatrix}$$

Select the most negative one : s_1

(2):

$$p_1 = c_1 - z_1 = 56;\ p_2 = c_2 - z_2 = 50$$

(row l of B^{-1})$N = [-0.3 - 0.3]$

$$\varepsilon = \min_j \left\{ \frac{z_j - c_j}{\alpha_{lj}} : \alpha_{lj} < 0 \right\} = \min\left[\frac{56}{0.3} \quad \frac{50}{0.3} \right] = \frac{500}{3}$$

Dual Simplex Algorithm Steps

(3):

$$\Rightarrow \text{column 2 comes into the basis} \Rightarrow \text{basis} \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$$

$$\text{or } \underline{\lambda}^T = \underline{\lambda}^T - \varepsilon(\text{row}_l \text{ of } (B^{-1})) = [0 \quad 0 \quad 0] - \frac{500}{3}[-1 \quad 0 \quad 0] = \begin{bmatrix} \frac{500}{3} & 0 & 0 \end{bmatrix}$$

$$\text{new } B = \begin{bmatrix} 0.3 & 0 & 0 \\ 0.4 & -1 & 0 \\ 0.2 & 0 & -1 \end{bmatrix} \quad \text{new } B^{-1} = \begin{bmatrix} 10/3 & 0 & 0 \\ 4/3 & -1 & 0 \\ 2/3 & 0 & -1 \end{bmatrix}$$

$$\underline{\lambda}^T = [50 \quad 0 \quad 0] \begin{bmatrix} 10/3 & 0 & 0 \\ 4/3 & -1 & 0 \\ 2/3 & 0 & -1 \end{bmatrix} = [500/3 \quad 0 \quad 0]$$

$$\underline{x}_B = \begin{bmatrix} 10/3 & 0 & 0 \\ 4/3 & -1 & 0 \\ 2/3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 900{,}000 \\ 800{,}000 \\ 500{,}000 \end{bmatrix} = \begin{bmatrix} 3{,}000{,}000 \\ 400{,}000 \\ 100{,}000 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$$

$$\Rightarrow \textit{Optimal} \Rightarrow f^* = \$150M$$

Key Idea of Primal-Dual Algorithm

- Idea for Primal-Dual Algorithm
  - To set the stage, consider the SLP and its dual

Primal		*Dual*
$\min \underline{c}^T \underline{x}$ $s.t.\ A\underline{x} = \underline{b}$ $\underline{x} \geq \underline{0}$	$\Leftrightarrow$	$\max \underline{\lambda}^T \underline{b}$ $s.t.\ \underline{\lambda}$ unrestricted $\underline{\lambda}^T A \leq \underline{c}^T$

  - At optimum:

    $\underline{\lambda}^T (A\underline{x} - \underline{b}) = 0\ldots$ satisfied for any feasible $\underline{x}$ in primal and

    $(\underline{c}^T - \underline{\lambda}^T A)\underline{x} = 0\ldots$ satisfied at optimum
  - Suppose we have a feasible $\underline{\lambda}$ for the dual problem

    $\Rightarrow \underline{\lambda}^T A \leq \underline{c}^T$

    $\Rightarrow$ Some of these inequalities will be equalities

    $\Rightarrow$ Define the subset P of $\{1,\ldots,n\}$ by $i \in P$

    $$P = \left\{ i : \underline{\lambda}^T \underline{a}_i = c_i \right\}$$

    If none, set $P = \varnothing$
  - For optimality, we need:

    $x_i > 0$ if $\underline{\lambda}^T \underline{a}_i = c_i \Rightarrow i \in P$

    $x_i = 0$ if $\underline{\lambda}^T \underline{a}_i < c_i \Rightarrow i \notin P$ $\Rightarrow$ so, if we can find x_i s.t. $x_i = 0$ for $i \notin P$, we are done!!

Maintaining Dual and Primal Feasibility

- What does it mean?
  - This amounts to searching for $\underline{x}$ such that

$$\sum_{i\in P} \underline{a}_i x_i = \underline{b} \qquad x_i \geq 0, i \in P\,; \quad x_i = 0, i \notin P$$

$\Rightarrow$ Nonnegative linear combinations of columns in P=$\underline{b}$

P = set of admissible columns

- But, this is simply phase I of LP … restricted primal (RP)

$$\min_{\underline{x},\underline{y}} \sum_{i=1}^{m} y_i = \underline{e}^T \underline{y} = \begin{bmatrix} \underline{0}^T & \underline{e}^T \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = \underline{c}^T \underline{x}_a$$

$$s.t. \quad \sum_{i\in P} \underline{a}_i x_i + \underline{y} = \underline{b}$$

$$x_i \geq 0, i \in P\,; x_i = 0, \ i \notin P \text{ (implicit)};\ \underline{y} \geq \underline{0}$$

- Dual of the restricted primal (DRP)

$$\max_{\underline{\mu}} \ \underline{\mu}^T \underline{b}$$

$$s.t. \quad \underline{\mu}^T \underline{a}_i \leq 0; i \in P$$

$$\underline{\mu} \leq \underline{e}$$

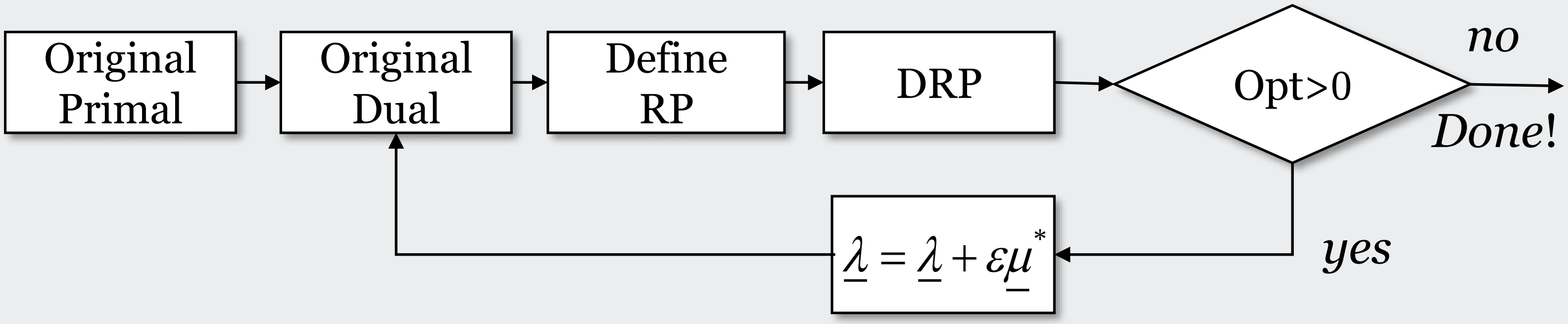
- Given a feasible $\underline{\lambda}$, we can find a feasible solution $\underline{x}$ to the associated RP
- If optimum solution of RP = 0, then found an optimum:

  $\underline{x}$ from RP & original $\underline{\lambda}$ are optimum

- Else, update $\underline{\lambda}$ via $\underline{\lambda} = \underline{\lambda} + \varepsilon \underline{\mu}^*$ where $\underline{\mu}^*$ = vector of simplex multipliers at the termination of RP

Primal-Dual Algorithm Graphically

- Graphically, the idea is this:



- Key questions
  - What is the sign of ε?
  - What is the largest ε I can take? … must maintain dual feasibility
  - Can I detect infeasibility?
  - Does the algorithm converge?
- Sign of ε
  - $\underline{\mu}^{*T}\underline{b} \geq 0$ since $\underline{\mu} = \underline{0}$ is feasible for DRP
  - New dual cost:

$$\underline{\lambda}^T\underline{b} = \underline{\lambda}^T\underline{b} + \varepsilon\underline{\mu}^{*T}\underline{b} = \underline{\lambda}^T\underline{b} + \varepsilon(\text{optimum solution of RP (or DRP)}) > \underline{\lambda}^T\underline{b} \text{ if } \varepsilon > 0$$

  - Must take $\varepsilon > 0$ to increase the cost of original dual

Step Size in Primal-Dual Algorithm

- Step size and detection of infeasibility
  - What is the effect of ε on feasibility?

    Need $\underline{\lambda}^{T}\underline{a}_i = \underline{\lambda}^T\underline{a}_i + \varepsilon\underline{\mu}^{*T}\underline{a}_i \le c_i \;\; \forall i = 1,\ldots,n$

    If $\underline{\mu}^{*T}\underline{a}_i < 0 \Rightarrow$ *No Problem*

    However, if $\underline{\mu}^{*T}\underline{a}_i < 0 \;\forall i$ then

    $\Rightarrow$ we can increase ε indefinitely, while maintaining dual feasibility

    $\Rightarrow$ dual is unbounded $\Rightarrow$ primal is infeasible
  - If optimal solution in RP>0 and the optimal dual satisfies $\underline{\mu}^{*T}\underline{a}_i < 0 \;\forall i \notin P$, then the original problem is infeasible (or original dual is unbounded)
  - If original problem has finite optimum
    - At least some $\underline{\mu}^{*T}\underline{a}_i > 0$ for $i \notin P$
    - ε should be chosen such that the equality is met by one of the constraints first

$$\varepsilon = \min_{i \notin P}\left\{\frac{c_i - \underline{\lambda}^T\underline{a}_i}{\underline{\mu}^{*T}\underline{a}_i} : \underline{\mu}^{*T}\underline{a}_i > 0\right\}$$

  - The dual cost increases to $\underline{\lambda}^{T}\underline{b} = \underline{\lambda}^T\underline{b} + \varepsilon\underline{\mu}^{*T}\underline{b}$
  - The set P changes to $P \leftarrow P \cup \{k\}$ where $k = \arg\min_{i \notin P}\left\{\frac{c_i - \underline{\lambda}^T\underline{a}_i}{\underline{\mu}^{*T}\underline{a}_i} : \underline{\mu}^{*T}\underline{a}_i > 0\right\}$

Primal-Dual Algorithm Steps

- **Primal-Dual Algorithm**

  Step 1:

  Given a feasible $\underline{\lambda}$ to the dual problem

  $$\max \underline{\lambda}^T \underline{b}$$
  $$s.t. \quad \underline{\lambda}^T A \le \underline{c}^T$$

  Determine the restricted primal problem:

  - Find set P
  - Formulate restricted primal: $\min \underline{e}^T \underline{y}$

    $$\text{s.t.} \quad \sum_{i \in P} \underline{a}_i x_i + \underline{y} = \underline{b}$$
    $$x_i \ge 0, i \in P; \quad x_i = 0, i \notin P \text{ (implicit)}; \ \underline{y} \ge \underline{0}$$

  - Note: $\underline{b} \ge \underline{0}$, if not, multiply corresponding Eq. by –1

  Step 2:

  Optimize the restricted primal (phase I of LP)

  If optimal solution = 0, then done

  Else go to Step 3

  Step 3:

  Compute $\underline{\mu}^{*T} \underline{a}_i$ for $i \notin P$

Illustration of Primal-Dual Algorithm

Step 3 (cont'd): If all $\underline{\mu}^{*T}\underline{a}_i < 0$ for $i \notin P$, then primal is infeasible

Else update $\underline{\lambda} \leftarrow \underline{\lambda} + \varepsilon\underline{\mu}^*$

Where $$\varepsilon = \frac{c_k - \underline{\lambda}^T\underline{a}_k}{\underline{\mu}^{*T}\underline{a}_k} = \min_{i \notin P}\left\{\frac{c_i - \underline{\lambda}^T\underline{a}_i}{\underline{\mu}^{*T}\underline{a}_i} : \underline{\mu}^{*T}\underline{a}_i > 0\right\}$$

$$P \leftarrow P \cup \{k\}$$

Go back to Step 1

Primal-Dual:

$$\begin{aligned} &\min 3x_1 + 4x_2 + 5x_3 \\ &s.t.\ x_1 + 2x_2 + 3x_3 \geq 5 \\ &\quad 2x_1 + 2x_2 + x_3 \geq 6 \\ &\quad x_i \geq 0 \end{aligned}$$

$$\begin{aligned} &\max 5\lambda_1 + 6\lambda_2 \\ &s.t.\ \lambda_1 + 2\lambda_2 \leq 3 \\ &\quad 2\lambda_1 + 2\lambda_2 \leq 4 \\ &\quad 3\lambda_1 + \lambda_2 \leq 5 \\ &\quad \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

Iteration 0:

Let $\underline{\lambda} = 0,\ \{c_i - \underline{\lambda}^T\underline{a}_i\} = [3\ \ 4\ \ 5] \Rightarrow P = \phi$

Restricted primal:

$RP : \min \underline{e}^T\underline{y}$ s.t. $\underline{y} = \underline{b}$; $\underline{y} \geq \underline{0}$

$DRP : \max \underline{\mu}^T\underline{b}$ s.t. $\underline{\mu} \leq \underline{e}$ $\Rightarrow$ $\underline{y} = \underline{b},\ \underline{\mu}^T = \underline{e}^T$

$\underline{\mu}^T\{\underline{a}_i\} = [3\ \ 4\ \ 4]$

$\varepsilon = \min\left[\frac{3}{3}\ \ \frac{4}{4}\ \ \frac{5}{4}\right]$ $\Rightarrow$ Both 1 & 2 can enter basis

$P = \{1,2\};\ \underline{\lambda}^T = \underline{\lambda}^T + \varepsilon\underline{\mu}^T = [0\ \ 0] + 1[1\ \ 1] = [1\ \ 1]$

Property of Primal-Dual Algorithm

Iteration 1:

RP:

$$\min \underline{e}^T \underline{y}$$
$$\text{s.t.} \begin{bmatrix}1\\2\end{bmatrix} x_1 + \begin{bmatrix}2\\2\end{bmatrix} x_2 + \underline{y} = \underline{b} = \begin{bmatrix}5\\6\end{bmatrix}$$

DRP:

$$\max 5\mu_1 + 6\mu_2$$
$$\text{s.t.} \quad \mu_1 + 2\mu_2 \le 0$$
$$2\mu_1 + 2\mu_2 \le 0 \qquad \Rightarrow \mu_1 = \mu_2 = 0$$
$$\mu_1 \le 1$$
$$\mu_2 \le 1$$
$$\Rightarrow \underline{\lambda}^T = [1 \quad 1]; \text{optimal basis}, B = \begin{bmatrix}1 & 2\\2 & 2\end{bmatrix}; B^{-1} = \begin{bmatrix}-1 & 1\\1 & \frac{-1}{2}\end{bmatrix}$$
$$\underline{x}_B = B^{-1}\underline{b} = [1 \quad 2]^T \qquad \Rightarrow \underline{x}^T = [1 \quad 2 \quad 0]$$

- **Property of primal-dual algorithm**
  - Every column $i \in P$ in the optimal basis of restricted primal (RP) remains in set *P* at the start of next iteration
  - Proof:
    - If a column *i* is in the optimal basis of RP, $(\underline{\mu}^*)^T \underline{a}_i = 0$

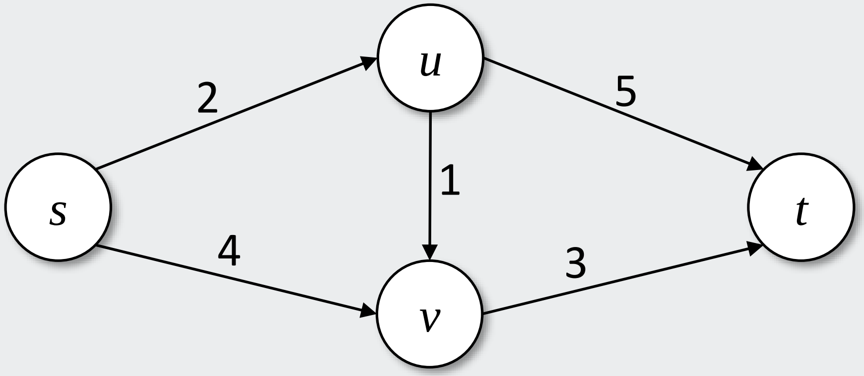
$$\Rightarrow \underline{\lambda}^T \underline{a}_i = \underline{\lambda}^T \underline{a}_i + \varepsilon \underline{\mu}^{*T} \underline{a}_i = \underline{\lambda}^T \underline{a}_i = c_i, \text{since } i \in P$$

- The algorithm must converge
  - No primal basis is repeated

Primal-Dual Algorithm for Shortest Path Problem

  - Pivoting on $\underline{a}_k$ will decrease restricted primal cost (since $(\underline{\mu}^*)^T \underline{a}_k > 0$)
  - There are only a finite number of bases

- Application to shortest path problem… Dijkstra's algorithm



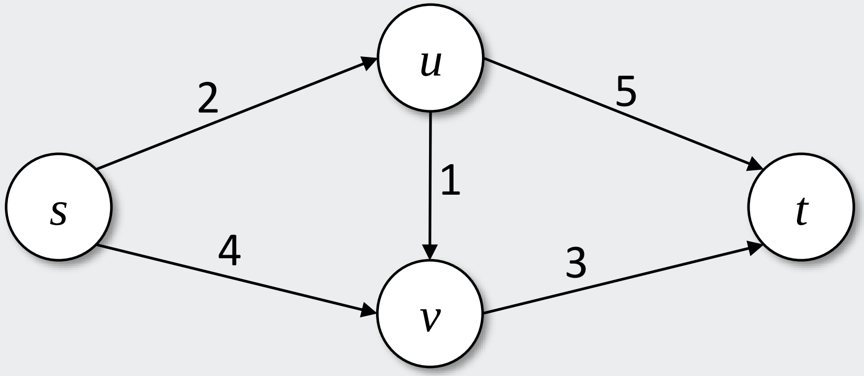
  - *s*, *u*, *v*, *t* are computers, edge lengths are costs of sending a message between them
  - Let x_{sv} be the fraction of messages sent from *s* to *v*

- **Primal**

$$\min \quad 2x_{su} + 4x_{sv} + x_{uv} + 5x_{ut} + 3x_{vt}$$
$$\text{s.t.} \quad x_{su}, x_{sv}, x_{uv}, x_{ut}, x_{vt} = 0 \text{ or } 1$$

Primal-Dual Algorithm for Shortest Path Problem

$$A\underline{x} = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{su} \\ x_{sv} \\ x_{uv} \\ x_{ut} \\ x_{vt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{b}$$



- **Dual**
  - λ_s = Price of a message at node *s* (buying or selling) = 0
  - λ_t = Price of a message at node *t* (buying or selling)

$$\begin{aligned} \max\ & \lambda_t \\ \text{s.t.}\ & \lambda_u \le 2 \\ & \lambda_v \le 4 \\ & \lambda_v - \lambda_u \le 1 \\ & \lambda_t - \lambda_u \le 5 \\ & \lambda_t - \lambda_v \le 3 \end{aligned}$$

- **Crude way**
  - Start with $\underline{\lambda}^T = [0\ \ 0\ \ 0]; P = \phi$

Primal-Dual Algorithm for Shortest Path Problem

$\Rightarrow$ RP has solution $\underline{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ Optimal cost=1

$$\text{Basis} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \underline{\mu}^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

because $\max \mu_t$ s.t. $\mu_u \le 1,\ \mu_v \le 1,\ \mu_t \le 1$

<u>Iteration 1</u>: $(\underline{\mu}^*)^T \underline{a}_i = [1 \quad 1 \quad 0 \quad 0 \quad 0]$ for $i \notin P$

$$\varepsilon = \arg\min_{i \notin P} \left\{ \frac{c_i - \underline{\lambda}^T \underline{a}_i}{\underline{\mu}^T \underline{a}_i} : \underline{\mu}^T \underline{a}_i > 0 \right\} = \min[2 \quad 4 \quad x \quad x \quad x]$$

$\Rightarrow$ pick column 1 to enter admissible column set $P \Rightarrow P\{1\}$

- Update $\underline{\lambda} \Rightarrow \underline{\lambda}^T = [0 \quad 0 \quad 0] + 2[1 \quad 1 \quad 1] = [2 \quad 2 \quad 2]$
- x_{su}=1
- Dual of RP $\max \mu_t$

$$\begin{aligned} s.t.\ \ \mu_u &\le 0 \\ \mu_v &\le 1 \\ \mu_t &\le 1 \end{aligned} \qquad \Rightarrow \underline{\mu}^* = [0 \quad 1 \quad 1]$$

Primal-Dual Algorithm for Shortest Path Problem

Iteration 2:

$$\overline{P} = \{2 \quad 3 \quad 4 \quad 5\}$$

$$(\underline{\mu}^*)^T \underline{a}_i = [1 \quad 1 \quad 1 \quad 0] \quad \text{for } i \notin P$$

$$\varepsilon = \min\left\{\frac{4-2}{1}, \frac{1}{1}, \frac{5-2}{1}\right\} = 1 \Rightarrow P = \{1,3\}$$

$$\Rightarrow \underline{\lambda}^T = [2 \quad 2 \quad 2] + 1[0 \quad 1 \quad 1] = [2 \quad 3 \quad 3]$$

$$\Rightarrow x_{uv} = 1$$

Iteration 3:

$$\overline{P} = \{2 \quad 4 \quad 5\}$$

$$\begin{aligned} &\max \mu_t \\ &\text{s.t. } \mu_u \le 0 \\ &\quad \mu_v - \mu_u \le 0 \Rightarrow \underline{\mu}^* = [0 \quad 0 \quad 1] \\ &\quad \mu_v, \mu_t \le 1 \end{aligned}$$

$$(\underline{\mu}^*)^T \underline{a}_i = [0 \quad 0 \quad 1]; i \notin P$$

$$\Rightarrow \varepsilon = \min\left\{\frac{3-0}{1}\right\} = 3$$

$$\Rightarrow \underline{\lambda}^T = [2 \quad 3 \quad 3] + 3[0 \quad 0 \quad 1] = [2 \quad 3 \quad 6]$$

$$\Rightarrow x_{vt} = 1$$

Iteration 4:

$$\begin{aligned} &\max \mu_t \\ &\text{s.t. } \mu_u \le 0 \\ &\quad \mu_v - \mu_u \le 0 \Rightarrow \underline{\mu}^* = 0 \Rightarrow \text{optimal} \\ &\quad \mu_t - \mu_v \le 0 \end{aligned}$$

There is a method to our madness

- Shortest path from $s - t$: $s \rightarrow u \rightarrow v \rightarrow t$
  - $s \rightarrow u = 2 = \lambda_u$
  - $s \rightarrow v = 3 = \lambda_v$
  - $s \rightarrow t = 6 = \lambda_t$
- There is a method to our madness .... Related to Dijkstra's Algorithm
  - $\underline{\mu}^*$ at stage i, where j columns (or arcs) are in the admissible set is defined as follows:

    $\underline{\mu}^* = 0$ for all nodes reachable by paths from source s using arcs in P

    $\underline{\mu}^* = 1$ for all other nodes
  - Iteration 1: Since P is empty $\underline{\mu}^*=[1\ 1\ 1]$
  - Iteration 2: Since P includes column 1 (arc(s, u)), $\underline{\mu}^*=[0\ 1\ 1]\ldots$
  - Iteration 3: Since P includes columns 1 and 3 (arcs (s,u), (u,v)), $\underline{\mu}^*=(0\ 0\ 1)$
  - Iteration 4: Since P includes columns 1,3 and 5 (arcs (s,u), (u,v) and (v,t)), $\underline{\mu}^*=(0\ 0\ 0)$
- What about step size ε?

$$\varepsilon = \min_{\text{arcs} \notin P}\{\text{cost of arc} - (\lambda_{\text{end node of arc}} - \lambda_{\text{start node of arc}})\}$$

  - Note: Denominator $(\underline{\mu}^*)^{\mathrm{T}} \underline{a}_i$ is always 1 or 0. Recall unimodularity of A
  - So consider arcs with $\mu^*_{\text{end node of arc}} - \mu^*_{\text{start node of arc}} > 0$ (in this case 1)

Relation to Dijkstra's Algorithm

- Since $\underline{\mu}^* = 0$ for all nodes reachable by s using arcs in P, λ_i for these nodes remains fixed from the time node i enters the feasible set P until the conclusion of the algorithm
  - Note the evolution of $\underline{\lambda}$

    $[0\ 0\ 0] \rightarrow [2\ 2\ 2] \rightarrow [2\ 3\ 3] \rightarrow [2\ 3\ 6]$
- If we let w be the set of nodes reachable through arcs in P, λ_i for these nodes remains constant till the end of the algorithm
- At each iteration, one node is added to w until w becomes the entire set of nodes $s \rightarrow (s, u) \rightarrow (s, u, v) \rightarrow (s, u, v, t)$
- Looks like we terminate in $(n - 1)$ steps where n is the number of nodes… with some streamlining, this is DIJKSTRA's algorithm…Lecture 6
- λ_u, λ_v and λ_t are the lengths of the shortest paths from start node s
- Interior Point Algorithms
- Three major types
  - The primal and primal-dual path following algorithms
  - Affine scaling algorithms
  - Potential Reduction Algorithms

Interior Point Methods

- Path following algorithms
  - Discuss not the original Interior point algorithm, but an equivalent (and more general) formulation based on **Barrier functions**

SLP

$$\min_{\underline{x}} \underline{c}^T \underline{x}$$
$$s.t.\ A\underline{x} = \underline{b}$$
$$\underline{x} \geq \underline{0}$$
optimal solution $\underline{x}^*$

$\Leftrightarrow$

Barrier

$$\min_{\underline{x}} f(\underline{x}, \mu) = \underline{c}^T \underline{x} - \mu \sum_{j=1}^{n} \ln x_j$$
$$s.t.\ A\underline{x} = \underline{b}$$
$$\mu > 0$$
optimal solution $\underline{x}^*(\mu)$

  - Key: $\underline{x}^*(\mu) \rightarrow \underline{x}^*$ as the Barrier parameter $\mu \rightarrow 0$
  - $\exists$ many variations of Barrier function formulations… we will discuss them later or see references

- Consider the general NLP

$$\min_{\underline{x}} f(\underline{x})$$
$$s.t.\ A\,\underline{x} = \underline{b}$$

  - Suppose $\underline{x}$ is feasible, then $\bar{\underline{x}} = \underline{x} + \alpha\,\underline{d}$, $\underline{d}$ ~ search direction
  - Pick α s.t. $A\bar{\underline{x}} = \underline{b}$ (new point is feasible) and $f(\underline{x}) < f(\bar{\underline{x}})$

Newton's Method for NLP

- What does Newton's method do for this problem?
  - Feasibility $\Rightarrow A\underline{\bar{x}} = A\underline{x} + \alpha A\underline{d} = 0 \Rightarrow A\underline{d} = 0$
  - Newton's method fits a quadratic to $f(\underline{x})$ at the current point and takes $\alpha = 1$

$$f(\underline{x}+\underline{d}) = f(\underline{x}) + \underline{g}^T\underline{d} + \frac{1}{2}\underline{d}^T H \underline{d}, \quad \text{where} \quad \underline{g} = \nabla f(\underline{x}); H = \nabla^2 f(\underline{x})$$

  - **Newton's method solves a quadratic problem to find $\underline{d}$** (⇒ a weighted least squares problem)
  - Consider

$$\begin{aligned}&\min_{\underline{d}}\ \underline{g}^T\underline{d} + \tfrac{1}{2}\underline{d}^T H \underline{d}\\ &s.t.\ A\underline{d} = \underline{0}\end{aligned} \quad \Rightarrow \quad \begin{aligned}&\min_{\underline{d}}\ \tfrac{1}{2}\left\| H^{\frac{1}{2}}\underline{d} - H^{\frac{1}{2}}\underline{g}\right\|_2^2\\ &s.t.\quad A\underline{d} = \underline{0}\\ &\qquad H^{\frac{1}{2}} \text{ symmetric square root}\end{aligned}$$

  - Define Lagrangian function:

$$L(\underline{d},\underline{\lambda}) = \underline{g}^T\underline{d} + \tfrac{1}{2}\underline{d}^T H \underline{d} - \underline{\lambda}^T A\underline{d}; \quad \underline{\lambda} \sim \text{ Lagrange multiplier}$$

  - Karush-Kuhn-Tucker necessary conditions of optimality:

$$\Rightarrow \frac{\partial L}{\partial \underline{d}} = 0 \Rightarrow \underline{g} + H\underline{d} - A^T\underline{\lambda} = \underline{0}$$

$$\Rightarrow \frac{\partial L}{\partial \underline{\lambda}} = 0 \Rightarrow -A\underline{d} = \underline{0}$$

KKT Conditions for the Barrier Problem

- Special NLP = Barrier formulation of LP:

$$\underline{g} = \nabla f(\underline{x}) = \underline{c} - \mu D^{-1}\underline{e} \text{ and } H = \nabla^2 f(\underline{x}) = \mu D^{-2}$$

where

$$D = \text{Diag}(x_j);\ j = 1, 2, \ldots, n$$

$$\underline{e} = [1 \quad 1 \quad 1 \quad \cdots \quad 1]^T$$

- Karush-Kuhn-Tucker conditions for special NLP are:

$$\mu D^{-2}\underline{d} + (\underline{c} - \mu D^{-1}\underline{e} - A^T\underline{\lambda}) = \underline{0}$$

$$A\underline{d} = \underline{0}$$

- So,

$$\underline{d} = \frac{-1}{\mu} D^2(\underline{c} - \mu D^{-1}\underline{e} - A^T\underline{\lambda}) \qquad (1)$$

- Using $A\underline{d} = \underline{0}$ in (1), we get

$$\underline{\lambda} = (AD^2A^T)^{-1}AD^2(\underline{c} - \mu D^{-1}\underline{e}) \qquad (2)$$

$$\text{or } \underline{\lambda} = (AD^2A^T)^{-1}A(D^2\underline{c} - \mu D\underline{e}) \qquad (3)$$

$$\underline{d} = [I - D^2A^T(AD^2A^T)^{-1}A](D\underline{e} - \frac{1}{\mu}D^2\underline{c}) \qquad (4)$$

Path Following Algorithm

- So, $\underline{\lambda}$ is the solution of weighted least square (WLS) problem:

$$\min_{\underline{\lambda}} \left\| D[\underline{c} - \mu D^{-1}\underline{e} - A^T \underline{\lambda}] \right\|_2^2$$

- Barrier function (Path following) Algorithm:
  - Choose a strictly feasible solution and constant $\mu > 0$
  - Let the tolerance parameter be ε and a parameter associated with the update of μ be σ

for $k = 0,1,\ldots.,k_{\max}$

let $D = \text{Diag}(x_j)$

Compute the solution $\underline{\lambda}$ to $(AD^2A^T)\underline{\lambda} = AD^2(\underline{c} - \mu D^{-1}\underline{e})$...WLS solution

let $\underline{p} = \underline{c} - A^T\underline{\lambda}$

$$\underline{d} = \frac{-D^2(\underline{p} - \mu D^{-1}\underline{e})}{\mu} = -\frac{(D^2\underline{p} - \mu D\underline{e})}{\mu}$$

$\underline{x} = \underline{x} + \underline{d}$

if $\underline{x}^T\underline{p} < \varepsilon \rightarrow$ stop : $\underline{x}$ is near-optimal solution... complementary slackness condition

else $\mu \leftarrow (1 - \frac{\sigma}{\sqrt{n}})\mu$

end if

end

$k_{max} \approx 50$
$\sigma \approx 1/4 - 1/6$

Finding a Feasible Point

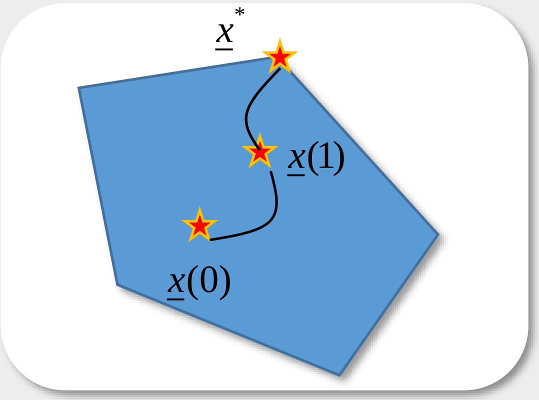


Illustration of Path Following Algorithms

- **Remarks**:
- Finding a feasible point

  Method 1

  - Select any $\underline{x}_0 > \underline{0}$ and define $\xi_0 \underline{y} = \underline{b} - A\underline{x}_0$ with $\|\underline{y}\|_2 = 1$

    $\Rightarrow \xi_0 = \|\underline{b} - A\underline{x}_0\|_2$ and solve:

$$
\begin{aligned}
&\min_{\underline{x},\xi} \xi \\
&s.t. \begin{bmatrix} A & \underline{y} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \xi \end{bmatrix} = \underline{b} \\
&\quad \underline{x} \geq \underline{0} \\
&\quad \xi \geq 0
\end{aligned}
$$

$$
\begin{aligned}
&Initial: \underline{x}_0 = \| \underline{b} \|_2 \, \underline{e} \\
&\xi = \| \underline{b} - A\underline{x}_0 \|_2 \\
&\underline{y} = \frac{\underline{b} - A\underline{x}_0}{\| \underline{b} - A\underline{x}_0 \|_2}
\end{aligned}
$$

Finding Feasible Point using M Method - 1

- The solution: $\xi = 0$ or when ξ starts becoming negative → stop
- Suggest $\underline{x}_0 = \|\underline{b}\|\underline{e}$

Method 2: … big M method

Primal	**Dual**
$\min \underline{c}^T \underline{x}$	$\max \underline{\lambda}^T \underline{b}$
s.t. $A\underline{x} = \underline{b}$	s.t. $\underline{\lambda}$ unrestricted
$\underline{x} \geq \underline{0}$	$\underline{\lambda}^T A \leq \underline{c}^T \Rightarrow \underline{\lambda}^T A + \underline{p}^T = \underline{c}^T$

- Assume A, $\underline{b}$ and $\underline{c}$ are integers with absolute values bounded by U (Can always do this by scaling numbers by 10^t, $t \sim 3-6$)
- Then,

$$\sum_{j=1}^{n} x_j = \underline{e}^T \underline{x} \leq n(mU)^m \text{ (very loose bound)}$$

- Let $\bar{\underline{b}} = \underline{b}(n+2)/n(mU)^m;\ x_i \leftarrow x_i(n+2)/n(mU)^m$

Finding Feasible Point using M Method - 2

- Finding a feasible point - Method 2 (cont'd...)

Primal

$$\begin{aligned}
&\min\ \underline{c}^T x + M x_{n+1} \\
&\text{s.t. } A\underline{x} + (\overline{b} - A\underline{e}) x_{n+1} = \overline{b} \\
&\underline{e}^T \underline{x} + x_{n+1} + x_{n+2} = n+2 \\
&\underline{x} \geq \underline{0} \\
&x_{n+1} \geq 0;\ x_{n+2} \geq 0
\end{aligned}$$

Dual

$$\begin{aligned}
&\max\ \underline{\lambda}^T \overline{b} + \lambda_{m+1}(n+2) \\
&\text{s.t. } \underline{\lambda}^T A + \lambda_{m+1}\underline{e}^T + \underline{p}^T = \underline{c}^T \\
&\underline{\lambda}^T \left(\overline{b} - A\underline{e}\right) + \lambda_{m+1} + p_{n+1} = M \\
&\lambda_{m+1} + p_{n+2} = 0 \\
&p_1, p_2, \ldots, p_{n+1}, p_{n+2} \geq 0
\end{aligned}$$

- If we let $\mu_0 = 4\sqrt{\|\underline{c}\|^2 + M^2}$

  $\left(\underline{x} \quad x_{n+1} \quad x_{n+2}\right)_0 = \left(\underline{e} \quad 1 \quad 1\right)$ and

  $\left(\underline{\lambda} \quad \lambda_{m+1} \quad \underline{p} \quad p_{n+1} \quad p_{n+2}\right) = \left(\underline{0} \quad -\mu_0 \quad \underline{c} + \mu_0\underline{e} \quad M + \mu_0 \quad \mu_0\right)$ are feasible solutions

- Since the method uses Newton's directions, expect quadratic convergence near minimum

Major Computational Step: WLS

- Major computational step: Weighted Least-squares subproblem

$$(AD^2A^T)\underline{\lambda} = AD^2(\underline{c} - \mu D^{-1}\underline{e})$$

  - Generally A is sparse
  - We will discuss the computational aspects of Least-squares subproblem later

- The algorithm (theoretically) requires $O(\sqrt{n}L)$ iterations with overall complexity $O(n^3L)$ where

$$L = \sum_{i=0}^{m}\sum_{j=1}^{n}\left[\log\left|a_{ij}\right| + 1\right] + 1$$

- In practice, the method typically takes 20 – 50 iterations even for very large problems (> 20,000 variables). Simplex, on the other hand, takes increasingly large numbers of iterations with the problem size n
- Initialize $\mu = 2^{O(L)}$ and $\sigma \approx \frac{1}{4}$ to $\frac{1}{6}$. In practice, we need to experiment with the parameters

Other Potential Functions

- Other potential functions:

$$f(\underline{x},\ q) = r\ln(\underline{c}^T\underline{x} - q) - \sum_j \ln x_j$$

where $r = n + \sqrt{n}$ and

q = a lower-bound on the optimal cost

- Problem with Barrier function approach:
  - Update of μ
  - Selection of initial μ and parameter σ

• **Dual Affine scaling**:

- Typically, the affine scaling methods are used on the dual problem

Primal		*Dual*		*Modified Dual*
$\min_{\underline{x}}\ \underline{c}^T\underline{x}$		$\max_{\underline{\lambda}}\ \underline{\lambda}^T\underline{b}$		$\max_{\underline{\lambda}}\ \underline{\lambda}^T\underline{b}$
s.t. $A\underline{x} = \underline{b}$	$\Leftrightarrow$	s.t. $A^T\underline{\lambda} \le \underline{c}$	$\Leftrightarrow$	s.t. $A^T\underline{\lambda} + \underline{p} = \underline{c}$
$\underline{x} \ge \underline{0}$				$\underline{p} \ge \underline{0}$

Dual problem and scaled reduced costs

- Suppose we have a strictly feasible $\underline{\tilde{\lambda}}$ and the corresponding reduced cost vector (slack vector) is $\underline{\tilde{p}}$
  - Define

$$\underline{\hat{p}} = P^{-1}\underline{p}$$

where

$$P = \text{Diag}\left[\, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n \right]$$

- So, the dual problem is:

$$\begin{aligned} &\max \underline{\lambda}^T \underline{b} \\ &\text{s.t.} \quad A^T \underline{\lambda} + P\underline{\hat{p}} = \underline{c} \\ &\underline{\hat{p}} \geq \underline{0} \end{aligned}$$

- From the equality constraint:

$$\begin{aligned} &\underline{\hat{p}} = P^{-1}(\underline{c} - A^T \underline{\lambda}) \\ &\Rightarrow P^{-1} A^T \underline{\lambda} = (P^{-1}\underline{c} - \underline{\hat{p}}) \end{aligned}$$

- Assuming full column rank of A^T or row rank of A

  $\Rightarrow$ linearly independent constraints in primal

LP for Scaled Reduced Costs

$$AP^{-2}A^T\underline{\lambda} = AP^{-1}(P^{-1}\underline{c} - \underline{\hat{p}})$$

$$\Rightarrow \underline{\lambda} = \left(AP^{-2}A^T\right)^{-1} AP^{-1}(P^{-1}\underline{c} - \underline{\hat{p}}) = M(P^{-1}\underline{c} - \underline{\hat{p}})$$

note that $\underline{\lambda} \in R(AP^{-1}) = R(M)$

- Eliminating $\underline{\lambda}$ from the dual problem we have:

$$\begin{aligned} &\max_{\underline{\hat{p}}} \ \underline{b}^T M(P^{-1}\underline{c} - \underline{\hat{p}}) = f(\underline{\hat{p}}) \\ &\text{s.t.} \quad H(\underline{\hat{p}} - P^{-1}\underline{c}) = \underline{0} \\ &\qquad \underline{\hat{p}} \geq \underline{0} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\min_{\underline{\hat{p}}} \ \underline{b}^T M \underline{\alpha} \\ &\text{s.t.} \ H\underline{\alpha} = \underline{0} \\ &\text{where } \underline{\alpha} = \underline{\hat{p}} - P^{-1}\underline{c} \end{aligned}$$

and where

$H = I - P^{-1}A^T M$, a symmetric projection matrix

$\Rightarrow H^2 = H$

- In addition, we have

$$AP^{-1}H = 0 \Rightarrow \text{columns of } H \in N\left(AP^{-1}\right)$$

Direction to Update Dual Variables

- Note that we want $\underline{\alpha} \in N(H) \Rightarrow \underline{\alpha} \in R(P^{-1}A^T)$
- But $R\left(P^{-1}A^T\right) = R\left(M^T\right)$
- The gradient of $f(\underline{\hat{p}})$ w.r.t. the scaled reduced costs $\underline{\hat{p}}$ is

$$\underline{\hat{g}}_p = -M^T\underline{b} \in R\left(M^T\right) = R\left(P^{-1}A^T\right)$$

$\Rightarrow$ **Results**: The gradient w.r.t. the scaled reduced costs, $\underline{\hat{p}}$, already lies in the range space of $P^{-1}A^T$... *making the projection unnecessary*

- In terms of the original unscaled reduced costs, the projected gradient is:

$$\underline{g}_p = P\underline{\hat{g}}_p = -A^T\left(AP^{-2}A^T\right)^{-1}\underline{b}$$

- The corresponding feasible direction with respect to $\underline{\lambda}$ is:

$$\underline{d}_\lambda = -MM^T\underline{\hat{g}}_p = \left(AP^{-2}A^T\right)^{-1}\underline{b}$$

$$\Rightarrow \quad \underline{g}_p = -A^T\underline{d}_\lambda$$

- If $\underline{g}_p \geq \underline{0} \Rightarrow$ dual problem is unbounded $\Rightarrow$ primal is infeasible (assuming $\underline{b} \neq \underline{0}$)

Dual Affine Scaling Algorithm Steps - 1

- Otherwise, we replace $\underline{\lambda}$ by

$$\underline{\lambda} \leftarrow \underline{\lambda} + \alpha \underline{d}_\lambda$$

where $\alpha = \beta\alpha_{\max;}$ $\beta \approx 0.95$

$$\alpha_{\max} = \min\left\{ \frac{-p_i}{g_{p_i}} : g_{p_i} < 0, i = 1, 2, \ldots, n \right\}$$

- Note that primal solution $\underline{x}$ is:

$$\underline{x} = -P^{-2}\underline{g}_p = P^{-2}A^T\left(AP^{-2}A^T\right)^{-1}\underline{b}$$

since it satisfies $A\underline{x} = \underline{b}$

- **Dual Affine Scaling Algorithm**:
  - Start with a strictly feasible $\underline{\lambda}$, stopping criterion ε and β

$z_{old} = \underline{\lambda}^T \underline{b}$

for $k = 0, 1, \ldots k_{\max}$

$p = \underline{c} - A^T\underline{\lambda}$

$P = \text{Diag}\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}$

Compute the solution $\underline{d}_\lambda$ to

$\left(AP^{-2}A^T\right)\underline{d}_\lambda = \underline{b}$

$\underline{g}_p = -A^T\underline{d}_\lambda$

Dual Affine Scaling Algorithm Steps - 2

if $\underline{g}_p \geq \underline{0}$

Stop $\rightarrow$ unbounded dual solution $\Rightarrow$ primal is infeasible

else

$$\alpha = \beta \min\left\{ \frac{-p_i}{g_{p_i}} : g_{p_i} < 0, i = 1, 2, \cdots, n \right\}$$

$$\underline{\lambda} \leftarrow \underline{\lambda} + \alpha \underline{d}_\lambda \left(\Rightarrow \underline{p} \leftarrow \underline{p} + \alpha \underline{g}_p \text{ next step} \right)$$

$$z_{new} = \underline{\lambda}^T \underline{b}$$

if $\dfrac{\left| z_{new} - z_{old} \right|}{\max\left(1, \left| z_{old} \right|\right)} < \varepsilon$

stop $\rightarrow$ found an optimal solution $\underline{x} = -P^{-2} \underline{g}_p$

else

$z_{old} \leftarrow z_{new}$

end if

end if

end do

UCONN

Initial Feasible Solution for Dual Affine Scaling Algorithm

- Finding an initial strictly feasible solution for the dual affine scaling algorithm

$$\underline{\lambda}_0 = \left(\frac{\|\underline{c}\|_2}{\|A^T \underline{b}\|_2} \right) \underline{b}$$

  - Want to find a $\underline{p}$ s.t. $\underline{p} = -\xi \underline{e}$
  - Select initial ξ_0 as

$$\xi_0 = -2 \min \left\{ \left(\underline{c} - A^T \underline{\lambda} \right)_i : i = 1, 2, \cdots, m \right\}$$

  - Solve an $(m+1)$ variable LP:

$$\max_{\underline{\lambda},\xi} \quad \underline{\lambda}^T \underline{b} - \mu \xi$$
$$\text{s.t.} \quad A^T \underline{\lambda} - \xi \underline{e} \le \underline{c}$$

  - Select $\mu = \gamma \dfrac{\underline{\lambda}_0^T \underline{b}}{\xi_0}; \quad \gamma = 10^5$
  - The initial $(\underline{\lambda}_0, \xi_0)$ are feasible for the problem
  - Note :
    - ❖ If $\xi < 0$ at iteration $k \Rightarrow$ found a feasible $\underline{\lambda}$
    - ❖ If the algorithm is such that optimal $\xi < \varepsilon \Rightarrow$ dual is infeasible $\Rightarrow$ primal is unbounded

Primal Affine Scaling

- Primal affine scaling
  - Starting with $\underline{x}_0 \to \underline{x}_1 \to \cdots \to \underline{x}_k \to \underline{x}_{k+1} \to \cdots \underline{x}^*$
  - $\underline{x}_{k+1} = \underline{x}_k + \underline{d}_k \ni \left\| D_k^{-1}\underline{d}_k \right\| \le \beta; \;\; \beta < 2/3; \;\; D_k = \text{Diag}(\underline{x}_k)$
  - $\underline{d}_k$ is the solution of

$$\begin{aligned} &\min \underline{c}^T \underline{d} \\ &\text{s.t. } A\underline{d} = \underline{0} \qquad\qquad \text{recall } A\underline{x} = \underline{b} \Rightarrow A\underline{d}_k = \underline{0} \\ &\quad \left\| D_k^{-1}\underline{d} \right\| \le \beta \end{aligned}$$

  - Lagrangian:

$$L(\underline{d}, \underline{\lambda}, \mu) = \underline{c}^T \underline{d} - \underline{\lambda}^T A\underline{d} + \tfrac{\mu}{2}(\underline{d}^T D_k^{-2}\underline{d} - \beta^2)$$

$$\Rightarrow \mu D_k^{-2}\underline{d} + \underline{c} - A^T\underline{\lambda} = \underline{0} \qquad \Rightarrow \underline{d} = -\tfrac{1}{\mu} D_k^2(\underline{c} - A^T\underline{\lambda})$$

$$A\underline{d} = \underline{0}$$

$$\underline{d}^T D_k^{-2}\underline{d} = \beta^2$$

$$\Rightarrow \tfrac{1}{\mu^2}(\underline{c} - A^T\underline{\lambda})^T D_k^2(\underline{c} - A^T\underline{\lambda}) = \beta^2$$

$$\Rightarrow \mu = \frac{\left\| D_k(\underline{c} - A^T\underline{\lambda}) \right\|_2}{\beta}$$

$$\Rightarrow \underline{\lambda}_k = (AD_k^2A^T)^{-1}AD_k^2\underline{c}; \quad \underline{d}_k = -\beta \frac{D_k^2(\underline{c} - A^T\underline{\lambda})}{\left\| D_k(\underline{c} - A^T\underline{\lambda}) \right\|_2}$$

Primal Affine Scaling Algorithm Steps

- Affine Scaling Algorithms

Start with $\underline{x}_0 > \underline{0}$

for $k = 0, 1, 2, \cdots k_{\max}$

$D_k = \text{Diag}(\underline{x}_k)$

$(AD_k^2 A^T)\underline{\lambda}_k = AD_k^2 \underline{c}$

$\underline{p}_k = \underline{c} - A^T \underline{\lambda}_k$

If $\underline{p}_k \geq \underline{0}$ and $\underline{e}^T D_k \underline{p}_k < \varepsilon$, stop $\rightarrow$ found optimal solution

else if $-D_k^2 \underline{p}_k \geq \underline{0} \Rightarrow$ primal is unbounded (cost $= -\infty$)

else

$$\underline{x}_{k+1} = \underline{x}_k - \beta \frac{D_k^2 \underline{p}_k}{\left\| D_k^2 \underline{p}_k \right\|_2}$$

end if

end

- Initialize via big-M method

Potential Reduction Algorithm

- Potential Reduction Algorithm

Primal	Dual
$\min\ \underline{c}^T \underline{x}$	$\max_{\underline{\lambda},\underline{p}}\ \underline{\lambda}^T \underline{b}$
s.t. $A\underline{x} = \underline{b}$	s.t. $\underline{\lambda}^T A + \underline{p}^T = \underline{c}^T$
$\underline{x} \geq \underline{0}$	$\underline{p}^T \geq \underline{0}$

- Modified Barrier Function $f(\underline{x}, \underline{p}) = q\ln(\underline{p}^T \underline{x}) - \sum_{j=1}^{n} \ln x_j - \sum_{j=1}^{n} \ln p_j$

  Note: $\underline{c}^T \underline{x} - \underline{\lambda}^T \underline{b} = (\underline{p}^T + \underline{\lambda}^T A)\underline{x} - \underline{\lambda}^T A\underline{x} = \underline{p}^T \underline{x}$

  Duality gap if $\underline{x}$ is primal feasible and $(\underline{\lambda}, \underline{p})$ are dual feasible

  Idea: Starting with $\underline{x}_k > 0$ and $\underline{p}_k \geq \underline{0}$, find a direction $\underline{d}_k$ such that

$$\min_{\underline{d}}\ \nabla \underline{f}_k^T \underline{d}$$

$$\text{s.t.}\quad A\underline{d} = 0$$

$$\left\| D_k^{-1}\underline{d} \right\| \leq \beta < 1$$

$$\nabla_{\underline{x}}\ \underline{f}_k = \frac{q}{\underline{p}_k^T \underline{x}_k}\underline{p}_k - D_k^{-1}\underline{e} = \underline{\hat{c}}$$

Solution:

$$\underline{d}_k = -\beta D_k \frac{\underline{u}}{\|\underline{u}\|}$$

$$\underline{u} = D_k\left(\underline{\hat{c}}_k - A^T\left(AD_k^2A^T\right)^{-1} AD_k^2\underline{\hat{c}}_k\right)$$

Potential Reduction Algorithm Steps

- Start with $\underline{x}_0 > 0,\ \underline{p}_0 > 0,\ \underline{\lambda}_0,\ \beta < 1,\ \gamma < 1,\ q$

for $k = 0,1,2,\ldots k_{\max}$

If $\underline{p}_k^T \underline{x}_k < \varepsilon$ stop, found optimal solution.

Else $\quad D_k = \text{Diag}(\underline{x}_k)$

$$\underline{\hat{c}}_k = \frac{q}{\underline{p}_k^T \underline{x}_k} \underline{p}_k - D_k^{-1} \underline{e}$$

$$\underline{u} = D_k \left(I - A^T \left(A D_k^2 A^T \right)^{-1} A D_k^2 \right) \underline{\hat{c}}_k; \quad \underline{d}_k = -\beta D_k \frac{\underline{u}}{\|\underline{u}\|}$$

If $\|\underline{u}\| \geq \gamma \Rightarrow$ perform primal step

$$\underline{x}_{k+1} = \underline{x}_k + \underline{d}_k$$

$$\underline{p}_{k+1} = \underline{p}_k$$

$$\underline{\lambda}_{k+1} = \underline{\lambda}_k$$

Else $\quad \underline{x}_{k+1} = \underline{x}_k$

$$\underline{p}_{k+1} = \frac{\underline{p}_k^T \underline{x}_k}{q} D_k^{-1} \left(\underline{u}_k + \underline{e} \right)$$

$$\underline{\lambda}_{k+1} = \underline{\lambda}_k + \left(A D_k^2 A^T \right)^{-1} A D_k \left(D_k \underline{p}_k - \frac{\underline{p}_k^T \underline{x}_k}{q} \underline{e} \right)$$

end if

end if

end

See page 415 of Bertsimas & Tsitsiklis

Primal-dual Path following Algorithms

- Primal-dual path following algorithms

Barrier formulation of primal

$$\min \; \underline{c}^T \underline{x} - \mu \sum_{j=1}^{n} \ln x_j$$
$$\text{s.t.} \quad A\underline{x} = \underline{b}$$

Barrier formulation of dual

$$\max_{\underline{\lambda}, \underline{p}} \; \underline{\lambda}^T \underline{b} + \mu \sum_{j=1}^{n} \ln p_j$$
$$\text{s.t.} \quad \underline{\lambda}^T A + \underline{p}^T = \underline{c}^T$$

- Optimality Conditions

$$\left.\begin{aligned}
&A\underline{x} = \underline{b}\\
&A^T\underline{\lambda} + \underline{p} = \underline{c}\\
&\underline{c} - \mu D^{-1} e - A^T\underline{\lambda} = \underline{0}\\
&\Rightarrow \underline{c} - \mu D^{-1} e - \underline{c} + \underline{p} = \underline{0}\\
&\Rightarrow \mu\underline{e} = D\underline{p} = DP\underline{e}\\
&P = \text{Diag}(\underline{p})
\end{aligned}\right\} \Rightarrow \begin{cases} A\underline{x} - \underline{b} = \underline{0}\\ A^T\underline{\lambda} + \underline{p} - \underline{c} = \underline{0}\\ DP\underline{e} - \mu\underline{e} = \underline{0}\end{cases}$$

- Nonlinear equation because of $Dp\underline{e} = \mu\underline{e}$ (complementary slackness condition when μ=0)

This is a nonlinear equation! We will revisit this issue later

Primal-dual Path following Algorithms

- Solve via Newton's Method

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ P_k & 0 & D_k \end{bmatrix}\begin{bmatrix} \underline{d}_x \\ \underline{d}_\lambda \\ \underline{d}_p \end{bmatrix} = -\begin{bmatrix} A\underline{x}_k - \underline{b} \\ A^T\underline{\lambda}_k + \underline{p}_k - \underline{c} \\ D_k P_k \underline{e} - \mu_k \underline{e} \end{bmatrix}$$

Basis of **infeasible** primal-dual method with $\underline{x}_k > \underline{0}$, $\underline{p}_k > \underline{0}$, and $\underline{\lambda}_k$

$$\Rightarrow \begin{cases} A\underline{d}_x = \underline{0} \\ A^T\underline{d}_\lambda + \underline{d}_p = \underline{0} \\ P_k\underline{d}_x + D_k\underline{d}_p = \mu_k\underline{e} - D_k P_k\underline{e} \end{cases}$$

Basis of **feasible** primal-dual method

- Solution:

$$\underline{d}_x = E_k\left(I - R_k\right)\underline{v}_k$$

$$\underline{d}_\lambda = -\left(AE_k^2A^{\mathrm{T}}\right)^{-1}AE_k\underline{v}_k$$

$$\underline{d}_p = E_k^{-1}P_k\underline{v}_k$$

where

$$E_k = D_k P_k^{-1}$$

$$R_k = E_k A^T\left(AE_k^2A^T\right)^{-1}AE_k$$

$$\underline{v}_k = D_k^{-1}E_k(\mu_k\underline{e} - D_kP_k\underline{e})$$

use $\mu_k = \dfrac{\underline{x}_k^T\underline{p}_k}{n}$

Primal-dual Path following Algorithm Steps

- Initialize

  $\underline{x}_0 > 0,\ \underline{p}_0 > 0,\ \underline{\lambda}_0, (\alpha < 1)$

for $k = 0, 1, 2, \ldots k_{\max}$

If $\underline{p}_k^T \underline{x}_k < \varepsilon$, stop

else (compute Newton directions)

$$\mu_k = \frac{\underline{x}_k^T \underline{p}_k}{n}$$

$$D_k = \mathrm{Diag}(\underline{x}_k)$$

$$P_k = \mathrm{Diag}(\underline{p}_k)$$

compute $\underline{d}_x, \underline{d}_\lambda$ and $\underline{d}_p$

find step lengths via

$$\beta_p = \min\left\{1, \alpha \min_{(i:d_{xi}<0)} \left(\frac{-x_{ki}}{d_{xi}}\right)\right\}$$

$$\beta_d = \min\left\{1, \alpha \min_{(i:d_{pi}<0)} \left(\frac{-p_{ki}}{d_{pi}}\right)\right\}$$

$$\underline{x}_{k+1} = \underline{x}_k + \beta_p \underline{d}_k$$

$$\underline{\lambda}_{k+1} = \underline{\lambda}_k + \beta_d \underline{d}_\lambda$$

$$\underline{p}_{k+1} = \underline{p}_k + \beta_d \underline{d}_p$$

end

Relationships among Path following Algorithms

- <u>Relationships</u>:
  - $\underline{d}_{affine} = -D^2\left(I - A^T\left(AD^2A^T\right)^{-1}AD^2\right)\underline{c}$
  - $\underline{d}_{primal\ path-following} = \left(I - D^2A^T\left(AD^2A^T\right)^{-1}A\right)\left(D\underline{e} - \frac{1}{\mu}D^2\underline{c}\right)$
  - When $\mu = \infty$, the corresponding direction is called *centering direction* because in this case $\underline{x}(\mu)$ is the *analytic center* of the feasible set.

$$\underline{d}_{centering} = \left(I - D^2A^T\left(AD^2A^T\right)^{-1}A\right)D\underline{e}$$

$$\Rightarrow \underline{d}_{primal\ path-following} = \underline{d}_{centering} + \frac{1}{\mu}\underline{d}_{affine}$$

$$\underline{d}_{potential} = \underline{d}_{centering} + \frac{q}{p^T\underline{x}}\underline{d}_{affine}$$

- Both potential and path following algorithms have polynomial complexity. There is no such result for affine scaling.
  - ⇒ centering directions are responsible for polynomiality of path following and potential reduction algorithms.

Implementation Issues

- Least-squares subproblem: Implementation Issues
  - Generally A is sparse
  - Major computational step at each iteration

    $AP^{-2}A^T\underline{d} = \underline{b}$ ⋯ Affine scaling

    $AD^2A^T\underline{\lambda} = AD^2(\underline{c} - \mu D^{-1}\underline{e}) = AD(D\underline{c} - \mu\underline{e})$ ⋯Barrier function method

    Similar equations in path following and potential reduction algorithms.
  - Key: Need to solve a symmetric positive definite system $\Sigma\underline{y} = \underline{b}$
- Solution Approaches:
- Direct methods:

  a) Cholesky factorization: $\Sigma = SS^T$, $S = \Delta_{lower}$

  b) ***LDL^T*** factorization: $\Sigma = LDL^T$; $L = \text{unit } \Delta_{lower}$

  c) ***QR*** factorization of $P^{-1}A^T$ or DA^T
- Methods to speed up factorization
  - During each iteration only D or P^{-1} changes, while A remains unaltered
    - Nonzero structure of Σ is static throughout
    - So, during the first iteration, keep track of the list of numerical operations performed

Factorization Methods

- Perform factorization only if the diagonal scaling matrix has changed significantly
  - Consider $\Sigma = AP^{-2}A^T$
  - Replace P by $\bar{P}$ where

$$\bar{P}_{ii}^{new} = \begin{cases} \bar{P}_{ii}^{old} & \text{if } \dfrac{\left|P_{ii} - \bar{P}_{ii}^{old}\right|}{\left|\bar{P}_{ii}^{old}\right|} < \delta \\ P_{ii} & \text{otherwise} \end{cases}$$

  - $\delta \sim 0.1$
  - Define $\Delta P_{ii} = \bar{P}_{ii}^{new} - \bar{P}_{ii}^{old}$
  - Then $\Sigma^{new} = \Sigma^{old} + \sum_{\{i:\Delta P_{ii} \neq 0\}} \Delta P_{ii} \underline{a}_i \underline{a}_i^T \quad \underline{a}_i = i^{th}$ column of A
  - So, use rank-one modification methods (ECE6435, Lecture 8)
- Perform pivoting to reduce fill-ins ⇒ having nonzero elements in factors where there are zero elements in Σ
  - Recall that $(P\Sigma P^T)P\underline{y} = P\underline{b}$
  - Unfortunately, finding the optimal permutation matrix to reduce fill-in is NP-complete
  - However, ∃ heuristics
    - ❖ Minimum degree
    - ❖ Minimum local fill-in

Incomplete Cholesky Algorithm

- Combine with an iterative method, if we have a few dense columns in A that will make impractically dense Σ (recall the outer product representation)
  - $\Rightarrow$ Hybrid factorization and conjugate gradient method called a preconditioned conjugate gradient method works well
- <u>Idea</u>: At iteration k, split columns of A into two parts $[S\ \bar{S}]$ where columns of A_s are sparse (i.e., have density < $\lambda(\approx 0.3)$)
  - Form $A_s P^{-2} A_s^T$
  - Find <u>incomplete</u> Cholesky factor L such that $Z_s = A_s P^{-2} A_s^T = LL^T$
  - Basically the idea is to step through the Cholesky decomposition, but setting $l_{ij} = 0$ if the corresponding $\Sigma_{s_{ik}} = 0$

<u>Incomplete Cholesky Algorithm</u>

for $k = 1, \ldots, m$ do
  $l_{kk} = \sqrt{\Sigma_{s_{kk}}}$
  for $i = k+1, \cdots, m$ do
    if $\Sigma_{s_{ik}} \neq 0$
      $l_{ik} = \frac{\Sigma_{s_{ik}}}{l_{kk}}$
    end if
  end do
  for $j = k+1, \ldots, m$ do
    for $i = j, \ldots, m$ do
      if $\Sigma_{s_{ij}} \neq 0$
        $\Sigma_{s_{ij}} = \Sigma_{s_{ij}} - l_{ik} l_{jk}$
      end if
    end do
  end do
end do

Conjugate Gradient Algorithm

- Now consider the original problem $\Sigma \underline{y} = A^T P^{-2} A \underline{y} = \underline{b}$

$$L^{-1}\Sigma\left(L^{-1}\right)^T L^T \underline{y} = L^{-1}\underline{b}$$

$$\Rightarrow Q\underline{u} = \underline{f}$$

where $\quad Q = L^{-1}\Sigma(L^{-1})^T;\ \underline{u} = L^T \underline{y};\ \underline{f} = L^{-1}\underline{b}$

- Solve $Q\underline{u} = \underline{f}$ via conjugate gradient algorithm … ECE6435
- Conjugate Gradient Algorithm:

$\underline{u} = \underline{f}$ …initial solution

$c = \|\underline{f}\|_2$ …norm of RHS

$\underline{r} = \underline{f} - Q\underline{u}$ …initial residual

(negative gradient of $\left(\frac{1}{2}\underline{u}^T Q\underline{u} - \underline{u}^T \underline{f}\right)$)

$p = \|\underline{r}\|_2^2$ …square norm of initial residual

$\underline{d} = \underline{r}$ …initial direction

$k = 0$

while $\frac{\sqrt{p}}{c} \geq \varepsilon$ and $k \leq k_{max}$ do

$\underline{\omega} = Q\underline{d}$

$\alpha = \frac{r}{\underline{d}^T Q\underline{d}} \cdots$ step length

$\underline{u} = \underline{u} + \alpha\underline{d} \cdots$ new solution

$\underline{r} = \underline{r} - \alpha\underline{w} \cdots$ new residual, $\underline{r} = \underline{f} - Q\underline{u}$

$\beta = \frac{\|\underline{r}\|_2^2}{p} \cdots$ parameter to update direction

$\underline{d} = \underline{r} + \beta\underline{d} \cdots$ new direction

$p = \|\underline{r}\|_2^2$

$k = k + 1$

end do

Computational load … $O(m^2 + 10m)$

Need to store only four vectors: $\underline{u}$, $\underline{r}$, $\underline{d}$ and $\underline{w}$

Mehrotra's Correction

Recall $Dp\underline{e} = \mu e$ is a nonlinear equation

$$DP\underline{e} = \mu_k \underline{e}$$
$$D = D_k + \Delta D_k ; P = P_k + \Delta P_k$$
$$\left(D_k + \Delta D_k\right)(P_k + \Delta P_k)\underline{e} = \mu_k \underline{e}$$

$$P_k \underline{d}_x + D_k \underline{d}_p = \mu_k \underline{e} - D_k P_k \underline{e} - \Delta D_k \Delta P_k \underline{e} = \mu_k \underline{e} - D_k P_k \underline{e} - \underline{d}_x \circ \underline{d}_p$$
$$\underline{d}_x \circ \underline{d}_p = \text{Hadamard Product} = [d_{x1} d_{p1} \; d_{x2} d_{p2} \; \text{........} d_{xn} d_{pn}]$$

Mehrotra's Correction: Solve for directions twice

1. Predictor step: First solve by setting $\underline{d}_x = \underline{d}_p = 0$ in RHS
2. Corrector step: Solve it again by plugging the values from step 1 in RHS

- Factorization makes this easy to implement
- Speeds up convergence

Simplex versus Interior Point Methods

- Comparison of simplex and dual affine scaling methods
  - Three types of test problems
- NETLIB test problems
  - 31 test problems
  - The library and test problem can be accessed via electronic mail: *netlib@anl-mcs* (ARPANET/CSNET) or research! netlib (UNIX network)
  - # of variables n ranged from 51 to 5533
  - # of constraints m ranged from 27 to 1151
  - # of non-zero elements in A ranged from 102 to 16276
  - Comparisons on IBM 3090

	Simplex	**Affine Scaling**
Iterations	(6,7157)	(19,55)
Ratio of time per iteration	(0.093, 0.356)	1
Total cpu time range (secs)	(0.01, 217.67)	(0.05, 31.70)
Ratio of cpu time (Simplex/Affine)	(0.2, 10.7)	1

Simplex versus Interior Point Methods

- Multi-commodity Network Flow problem
  - Specialized LP algorithms exist that are better than simplex
  - ∃ a program to generate random multi-commodity network flow problem called MNETGN
  - 11 problems were generated
  - # of variables $n \in (2606,8800)$
  - # of constraints $m \in (1406,4135)$
  - Non-zero elements in A ranged from 5212 to 22140

	Simplex	Specialized Simplex	
	MINOS 4.0	MCNF 85	Affine Scaling
Total # of iterations	(940, 21915)	(931, 16624)	(28, 35)
Ratios of time per iteration (w.r.t. Affine Scaling)	(0.010, 0.069)	(0.0018, 0.0404)	1
Total CPU time (secs)	(12.73, 1885.34)	(7.42, 260.44)	(6.51, 309.50)
Ratios of CPU times w.r.t. Affine Scaling	(1.96, 11.56)	(0.59, 4.15)	1

Simplex versus Interior Point Methods

- Timber Harvest Scheduling problems
  - 11 timber harvest scheduling problems using a program called FOR-PLAN
  - # of variables ranged from 744 to 19991
  - # of constraints ranged from 55 to 316
  - Non-zero elements in *A* ranged from 6021 to 176346

	Simplex (MINOS 4.0) Default Pricing	**Affine Scaling**
Total # of iterations	(534, 11364)	(38,71)
Ratio of time per iteration	(0.0141, 0.2947)	1
Total CPU time (secs)	(2.74, 123.62)	(0.85, 43.80)
Ratios of CPU times	(1.52, 5.12)	1

Summary and References

- Promising approach for large real-world LP problems
- Summary
  - Reviewed duality
  - Dual simplex and primal-dual algorithm
  - Interior point methods
    - Path following (primal, primal-dual)
    - Affine scaling
    - Potential reduction
- References

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