Solution 1

Prof. Krishna R. Pattipati

Dept. of Electrical and Computer Engineering
University of Connecticut
Contact: krishna@engr.uconn.edu (860) 486-2890

ECE 6108
Linear Programming and Network Flows

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- Problem 1

$$B_{new} = B + (\underline{a} - \underline{b}_i)\underline{e}_i^T$$

- Problem 9

$$B = LU \Rightarrow U = L^{-1}B$$

$$B_{new} = [\underline{b}_1 \underline{b}_2, \ldots., \underline{b}_{i-1}, \underline{a}, \underline{b}_{i+1} \ldots \underline{b}_n]$$

In this case, it is best to move column $\underline{a}$ to the end.

This can be done via a permutation matrix on the right.

This corresponds to exchanging the variables associated the columns.

$$\hat{B}_{new} = [\underline{b}_1 \underline{b}_2, \ldots., \underline{b}_{i-1}, \underline{b}_{i+1} \ldots \underline{b}_n, \underline{a}]$$

$$L^{-1}\hat{B}_{new} = [\underline{u}_1 \underline{u}_2, \ldots., \underline{u}_{i-1}, \underline{u}_{i+1} \ldots \underline{u}_n, L^{-1}\underline{a}] = H$$

$H =$ upper Hessenberg matrix

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$$H = L^{-1}\hat{B}_{new}$$

$$H = \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}; i = 2$$

H can be made upper triangular by applying elementary transformations of the form (for $j = i, i+1, .., n\text{-}1$):

$$M_j = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -m_{j+1,j} & 1 & \\ & & & 1 \end{bmatrix}; m_{j+1,j} = \frac{h_{j+1,j}}{h_{jj}}$$

$$\Rightarrow \hat{U} = M_{n-1} \ldots\ldots M_i H$$

$$\Rightarrow \hat{B}_{new} = LM_i^{-1} \ldots. M_{n-1}^{-1}\hat{U} = \hat{L}\hat{U}$$

computational load $= O((n-i)^2 + n^2) \Rightarrow O(n^2)$ since $1 \leq i \leq n$.

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- Problem 10

$$B = QR \Rightarrow R = Q^T B$$

$$\hat{B}_{new} = [\underline{b}_1 \underline{b}_2, \ldots, \underline{b}_{i-1}, \underline{b}_{i+1} \ldots \underline{b}_n, \underline{a}]$$

$$Q^T \hat{B}_{new} = [\underline{r}_1 \underline{r}_2, \ldots, \underline{r}_{i-1}, \underline{r}_{i+1} \ldots \underline{r}_n, Q^T \underline{a}] = H$$

H = upper Hessenberg matrix

Apply Givens transformations $J^T(j, j+1)$ for $j = i, i+1, .., n-1$ so that

$$J^T(n-1, n) \ldots J^T(i, i+1) H = \hat{R} \Rightarrow J^T(n-1, n) \ldots J^T(i, i+1) Q^T \hat{B}_{new} = \hat{R}$$

$$\Rightarrow \hat{Q} = Q J(i, i+1) \ldots J(n-1, n)$$

computational load $= O(n^2)$

$$J(i, i+1) = \begin{bmatrix} 1 & & & \\ & c & s & \\ & -s & c & \\ & & & 1 \end{bmatrix}; i = 2$$

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Problem 2

a) $f_1(x) = x \Rightarrow convex;\ f_2(x) = x^2 \Rightarrow convex$

$f_1(x)\, f_2(x) = x^3 \Rightarrow not\ convex$

$Other:\ f_1(x) = x \Rightarrow convex;\ f_2(x) = -x \Rightarrow convex$

$f_1(x)\, f_2(x) = -x^2 \Rightarrow concave$

$f_1(x) = e^{-x} \Rightarrow convex;\ f_2(x) = e^{x} - 1 \Rightarrow convex$

$g = f_1(x)\, f_2(x) = 1 - e^{-x} \Rightarrow No$

$\nabla^2 g = f_1 \nabla^2 f_2 + f_2 \nabla^2 f_1 + \nabla f_1 \nabla f_2$

$Need:\ f_1 > 0;\ f_2 > 0; \nabla f_1 \nabla f_2 > 0$

b) (i) $f(x_1, x_2) = x_1 x_2 \Rightarrow \nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow indefinite \Rightarrow not\quad convex$

(ii) $f(x_1, x_2) = e^{x_1 + x_2} \Rightarrow \nabla^2 f(x_1, x_2) = e^{x_1 + x_2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow convex\ but\ not\ strictly\ convex$

(iii) $f(x) = \tan x_1, 0 < x < 1 \Rightarrow \nabla^2 f(x) = 2 \tan x \sec^2 x > 0\ for\ 0 < x < 1 \Rightarrow strictly\ convex$

(iv) $f(x_1, x_2) = e^{-x_1 - x_2} + x_1^2 - 2x_1 \Rightarrow \nabla^2 f(x_1, x_2) = e^{-x_1 - x_2} \begin{bmatrix} 1 + 2e^{x_1 + x_2} & 1 \\ 1 & 1 \end{bmatrix} > 0 \Rightarrow strictly\ convex$

(v) $f = \max(f_1, f_2)\ where\ f_1 = x_1^2 + x_2^2;\ f_2 = x_1^2 - x_2$

f_1 is strictly convex; f_2 is convex.

If f_1 and f_2 are convex, $\max(f_1, f_2)$ *is convex. why?*

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- Problem 2

If f_1 *and* f_2 *are* convex, max(f_1, f_2) *is convex. why*?

$$f_1(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \le \alpha f_1(\underline{x}_1) + (1-\alpha) f_1(\underline{x}_2)$$

$$f_2(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \le \alpha f_2(\underline{x}_1) + (1-\alpha) f_2(\underline{x}_2)$$

$$\begin{aligned} f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) &= \max(f_1(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2), f_2(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2)) \\ &\le \max(\alpha f_1(\underline{x}_1) + (1-\alpha) f_1(\underline{x}_2), \alpha f_2(\underline{x}_1) + (1-\alpha) f_2(\underline{x}_2)) \\ &\le \alpha \max(f_1(\underline{x}_1), f_2(\underline{x}_1)) + (1-\alpha)\max(f_1(\underline{x}_2), f_2(\underline{x}_2)) \\ &= \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2) \end{aligned}$$

- Problem 3

a) If $f(\underline{x})$ is convex, $f(A\underline{x}+\underline{b})$ is convex.

$$f(\alpha(A\underline{x}_1 + \underline{b}) + (1-\alpha)(A\underline{x}_2 + \underline{b})) \le \alpha f(A\underline{x}_1 + \underline{b}) + (1-\alpha) f(A\underline{x}_2 + \underline{b})$$

Alternatively, let $\underline{y} = A\underline{x} + \underline{b} \Rightarrow f(\underline{y}) = f(A\underline{x} + \underline{b})$

$$\nabla_y f(\underline{y}) = A\nabla_x f(\underline{x}) \Rightarrow \nabla^2_{yy} f(\underline{y}) = A\nabla^2_{xx} f(\underline{x}) A^T \Rightarrow convex$$

b) $g(x_1) = f(x_1, x_2, \ldots, x_n)\,|_{x_2,\ldots,x_n = fixed}$

$$\begin{aligned} g(\alpha x_1^{'} + (1-\alpha)x_1^{''}) &= f(\alpha x_1^{'} + (1-\alpha)x_1^{''}, x_2, \ldots, x_n) \\ &\le \alpha f(x_1^{'}, x_2, \ldots, x_n) + (1-\alpha) f(x_1^{''}, x_2, \ldots, x_n) \\ &= \alpha g(x_1^{'}) + (1-\alpha) g(x_1^{''}) \end{aligned}$$

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- Problem 4

$f(\underline{x}) = \underline{x}^T Q \underline{x}$ is convex if Q is PD.

$$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = [\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2]^T Q [\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2]$$

$$= \alpha^2 \underline{x}_1^T Q \underline{x}_1 + 2\alpha(1-\alpha)\underline{x}_1^T Q \underline{x}_2 + (1-\alpha)^2 \underline{x}_2^T Q \underline{x}_2$$

$$\alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2) = \alpha \underline{x}_1^T Q \underline{x}_1 + (1-\alpha)\underline{x}_2^T Q \underline{x}_2$$

$$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) - \alpha f(\underline{x}_1) - (1-\alpha) f(\underline{x}_2)$$

$$= \alpha(\alpha-1)(\underline{x}_1^T Q \underline{x}_1 - 2\underline{x}_1^T Q \underline{x}_2 + \underline{x}_2^T Q \underline{x}_2)$$

$$= \alpha(\alpha-1)(\underline{x}_1 - \underline{x}_2)^T Q (\underline{x}_1 - \underline{x}_2) < 0 \text{ if } Q > 0 \text{ and } \underline{x}_1 \neq \underline{x}_2$$

- Problem 5

$$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = \sum_{i=1}^{n} f_i(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2)$$

$$\leq \sum_{i=1}^{n} \alpha f_i(\underline{x}_1) + (1-\alpha) f_i(\underline{x}_2) = \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$$

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- Problem 6

$$H = \{\underline{x} : \underline{c}^T \underline{x} = k\} \text{ is convex. Suppose } \underline{c}^T \underline{x}_1 = \underline{c}^T \underline{x}_2 = k$$

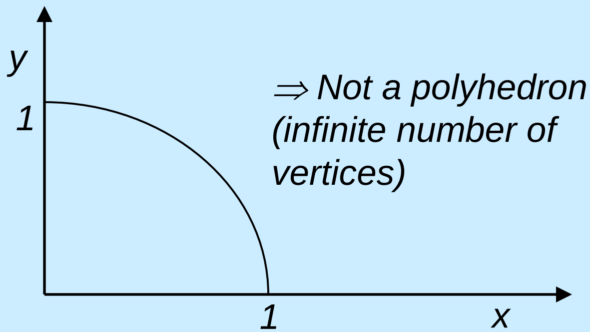
$$\underline{c}^T (\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = \alpha \underline{c}^T \underline{x}_1 + (1-\alpha)\underline{c}^T \underline{x}_2 = k$$

$$H^+ = \{\underline{x} : \underline{c}^T \underline{x} \le k\} \text{ is convex. Suppose } \underline{c}^T \underline{x}_1 = \underline{c}^T \underline{x}_2 \le k$$

$$\underline{c}^T (\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = \alpha \underline{c}^T \underline{x}_1 + (1-\alpha)\underline{c}^T \underline{x}_2 \le k$$

- Problem 7

$$(i)\{(x,y):(x,y) > 0 \; and \; x\cos\theta + y\sin\theta \le 1 \,\forall\, \theta \in [0, \pi/2]\}$$



⇒ Not a polyhedron (infinite number of vertices)

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- Problem 7

$$(ii)\ x^2-8x+15\le 0 \Rightarrow (x-3)(x-5)\le 0$$

$$\Rightarrow x\in[3,5] \Rightarrow \textit{polyhedron}$$

$$(iii)\ \text{consider}\ \{x \mid x\le 0, x\ge 1\} = \textit{empty set}$$

$$\Rightarrow \text{intersection of convex sets} \Rightarrow \text{polyhedron}$$

> •Some authors (e.g., Luenberger) let polytopes be empty and define polyhedrons as strictly non-emptty.

- Problem 8

$$\text{consider}\ \underline{x}_1\in s \text{ and } \underline{x}_2\in s \ni f(\underline{x}_1)\le c \text{ and } f(\underline{x}_2)\le c$$

$$\text{Then } f(\alpha\underline{x}_1+(1-\alpha)\underline{x}_2)\le \alpha f(\underline{x}_1)+(1-\alpha)f(\underline{x}_2)\le \alpha c+(1-\alpha)c=c$$

$$\Rightarrow \alpha\underline{x}_1+(1-\alpha)\underline{x}_2\in s$$

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