Solution 2

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ECE 364
Linear Programming and Network Flows

- Problem 1

$$Let\ \underline{a} = [a_0\ a_1 \ldots.. a_{n-1}]^T ; \underline{q} = [Q_1 Q_2 \ldots.. Q_m]^T$$

$$Let\ T = \begin{bmatrix} 1 & t_1 & . & t_1^{n-1} \\ 1 & t_2 & . & t_2^{n-1} \\ . & . & . & . \\ 1 & t_m & . & t_m^{n-1} \end{bmatrix} = \begin{bmatrix} \underline{t}_1^T \\ \underline{t}_2^T \\ . \\ \underline{t}_m^T \end{bmatrix}$$

$$(i)\ \min_{\underline{a}} \| T\underline{a} - \underline{q} \|_1 = \min_{\underline{a}} \sum_{i=1}^{m} | \underline{t}_i^T \underline{a} - Q_i | = \min_{\underline{a}} \sum_{i=1}^{m} (u_i + v_i)$$

Then the LP formulation is:

$$\min_{\underline{a}\geq \underline{0}, \underline{u}\geq \underline{0}, \underline{v}\geq \underline{0}} \underline{e}^T (\underline{u} + \underline{v}); \underline{e}^T = [11 \ldots. 1]$$

$$T\underline{a} - \underline{u} + \underline{v} = \underline{q}$$

$$\underline{e}^T \underline{a} = 1$$

$$| \underline{t}_i^T \underline{a} - Q_i | = z_i ; i = 1, 2, .., n$$

Then the LP formulation is:

$$\min_{\underline{a}\geq \underline{0}, \underline{z}\geq \underline{0}} \underline{e}^T \underline{z}; \quad \underline{e}^T = [11 \ldots. 1]$$

$$T\underline{a} - \underline{q} \leq \underline{z}$$

$$-T\underline{a} + \underline{q} \leq \underline{z}$$

$$\underline{e}^T \underline{a} = 1$$

- Problem 1

$$(ii)\ \min_{\underline{a}} \| T\underline{a} - \underline{q} \|_\infty = \min_{\underline{a}} \max_{1 \le i \le m} | \underline{t}_i^T \underline{a} - Q_i | = \min_{\underline{a}} w$$

$$\Rightarrow -w \le \underline{t}_i^T \underline{a} - Q_i \le w \quad \text{for } i = 1,2,..,m$$

Then the LP formulation is :

$$\min_{\underline{a} \ge \underline{0},\, w \ge 0} w$$

$$\begin{bmatrix} T & -\underline{e} \\ -T & -\underline{e} \\ \underline{e}^T & 0 \end{bmatrix} \begin{bmatrix} \underline{a} \\ w \end{bmatrix} \begin{bmatrix} \le q \\ \le -q \\ = 1 \end{bmatrix} \Rightarrow \text{can convert this to SLP}$$

$$(iii)\ \min_{\underline{a} \ge \underline{0}} \frac{1}{2} \| T\underline{a} - \underline{q} \|_2^2 = \min_{\underline{a} \ge \underline{0}} \frac{1}{2} \underline{a}^T T^T T \underline{a} - \underline{a}^T T^T \underline{q} + \frac{1}{2} \underline{q}^T \underline{q}$$

subject to: $\underline{e}^T \underline{a} = 1$

This is a quadratic programming problem.

- Problem 1

$$L(\underline{a},\lambda,\underline{\mu}) = \min_{\underline{a}\geq\underline{0},\underline{\mu}\geq\underline{0}}\left[\underline{a}^T T^T T\underline{a} - 2\underline{a}^T T^T \underline{q} + \underline{q}^T \underline{q} + \lambda(\underline{e}^T \underline{a} - 1) - \underline{\mu}^T \underline{a}\right]$$

Necessary conditions:

$$2T^T T\underline{a} - 2T^T \underline{q} + \lambda\underline{e} - \underline{\mu} = \underline{0} \Rightarrow \underline{a} = \left[T^T T\right]^{-1}\left[T^T \underline{q} - \frac{1}{2}\lambda\underline{e} + \frac{1}{2}\underline{\mu}\right]$$

$$\underline{e}^T \underline{a} = 1 \Rightarrow \lambda = \frac{\underline{e}^T\left[T^T T\right]^{-1}[2T^T \underline{q} + \underline{\mu}]}{\underline{e}^T\left[T^T T\right]^{-1}\underline{e}}$$

$$\text{Better way: add } -\mu\sum_{i=1}^{n}\ln a_i \text{ as penalty}$$

$$\mu_i a_i = 0, i = 0,1,2,..,n-1$$

Need a bound on x_2

- Problem 2

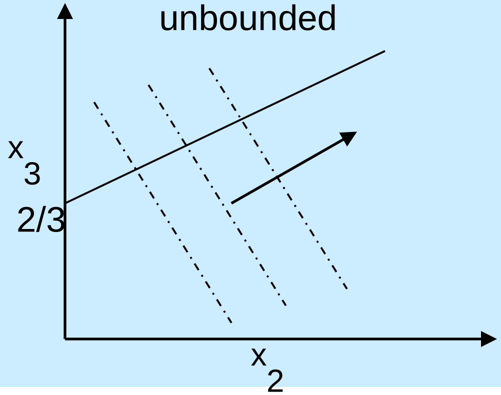
$$\max\ x_1 + 4x_2 + x_3$$
$$s.t.\ 2x_1 - 2x_2 + x_3 = 4$$
$$x_1 - x_3 = 1$$
$$x_2 \geq 0; x_3 \geq 0$$

$$\max\ 4x_2 + 2x_3$$
$$s.t. -2x_2 + 3x_3 = 2$$
$$x_2 \geq 0; x_3 \geq 0$$

$$\max\ 8x_3 \quad s.t.\ x_3 \geq 0$$

or

$$\max\ \frac{16}{3}x_2 \quad \text{s.t.}\ x_2 \geq 0$$



Problem 2

$\min -x_1 - 4x_2 - x_3$

$$\underline{x}_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 2/3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underline{\lambda}^T = \underline{c}_B^T B^{-1} = [-1\,-1]\begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} = [-2/3\; 1/3]$$

Reduced costs:

$$p_1 = c_1 - \underline{\lambda}^T \underline{a}_1 = -1 - [-2/3\; 1/3]\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

$$p_2 = c_2 - \underline{\lambda}^T \underline{a}_2 = -4 - [-2/3\; 1/3]\begin{bmatrix} -2 \\ 0 \end{bmatrix} = -8/3$$

$$p_3 = c_3 - \underline{\lambda}^T \underline{a}_3 = -1 - [-2/3\; 1/3]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Bring $\underline{a}_2$ into the basis

$$\underline{\alpha} = B^{-1}\underline{a}_2 = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ *unbounded*

Primal unbounded; Dual infeasible

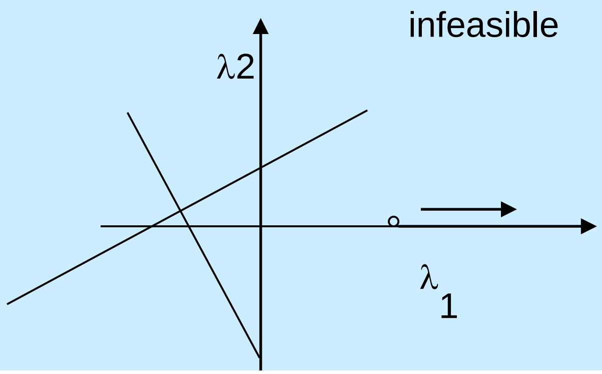
Dual:

$$\begin{aligned} &\max\; 4\lambda_1 + \lambda_2 \\ &s.t.\; 2\lambda_1 + \lambda_2 = -1 \\ &\quad -2\lambda_1 \le -4 \Rightarrow \lambda_1 \ge 2 \\ &\quad \lambda_1 - \lambda_2 \le -1 \end{aligned}$$

$\Rightarrow$

Dual:

$$\begin{aligned} &\max\; 2\lambda_1 \\ &s.t.\; \lambda_1 \ge 2 \\ &\quad \lambda_1 \le -2/3 \end{aligned}$$



Problem 2

$$\min -x_1 - 4x_2 - x_3$$

$$\underline{x}_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 2/3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underline{\lambda}^T = \underline{c}_B^T B^{-1} = [-1\ -1]\begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} = [-2/3\ 1/3]$$

Reduced costs:

$$p_1 = c_1 - \underline{\lambda}^T \underline{a}_1 = -1 - [-2/3\ 1/3]\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

$$p_2 = c_2 - \underline{\lambda}^T \underline{a}_2 = -4 - [-2/3\ 1/3]\begin{bmatrix} -2 \\ 0 \end{bmatrix} = -8/3$$

$$p_3 = c_3 - \underline{\lambda}^T \underline{a}_3 = -1 - [-2/3\ 1/3]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Bring $\underline{a}_2$ *into the basis*

$$\underline{\alpha} = B^{-1}\underline{a}_2 = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ *unbounded*

Primal unbounded; Dual infeasible

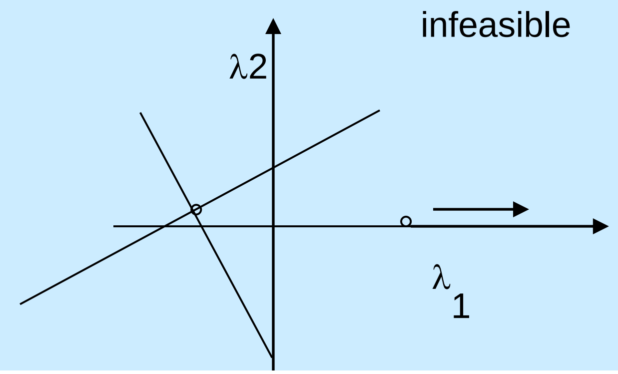
Dual:

$$\max 4\lambda_1 + \lambda_2$$
$$s.t.\ 2\lambda_1 + \lambda_2 = -1$$
$$-2\lambda_1 \le -4 \Rightarrow \lambda_1 \ge 2$$
$$\lambda_1 - \lambda_2 \le -1$$

$\Rightarrow$

Dual:

$$\max 2\lambda_1$$
$$s.t.\ \lambda_1 \ge 2$$
$$\lambda_1 \le -2/3$$



Problem 3

$x_1 =$ number of barrels of light crude

$x_2 =$ number of barrels of heavy crude

$\min 56x_1 + 50x_2$

$s.t.\ 0.3x_1 + 0.3x_2 \geq 900{,}000$

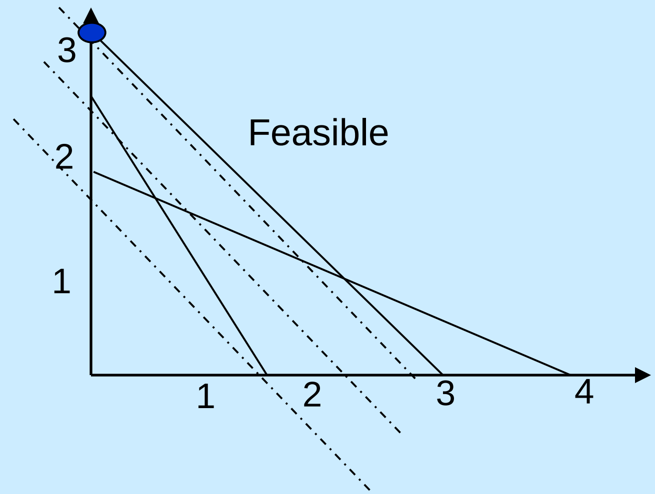
$\quad 0.2x_1 + 0.4x_2 \geq 800{,}000$

$\quad 0.3x_1 + 0.2x_2 \geq 500{,}000$

$\quad x_1 \geq 0; x_2 \geq 0$

$optimal\ point: (0, 3M)$

$Cost: \$150M$



Dual: use dual simplex. Primal is easier here!

$\max 100{,}000[9\lambda_1 + 8\lambda_2 + 5\lambda_3]$

$s.t.\ 0.3\lambda_1 + 0.2\lambda_2 + 0.3\lambda_3 \leq 56$

$\quad 0.3\lambda_1 + 0.4\lambda_2 + 0.2\lambda_3 \leq 50$

$s.t.\ \lambda_1 \geq 0; \lambda_2 \geq 0; \lambda_3 \geq 0$

$optimal\ point: (500/3\ \ 0\ \ 0)$

$Cost: \$150M$

- Problem 4:

Let u_i = number of units produced using *regular* production in month i

v_i = number of units produced using *overtime* production in month i

x_i = number of units on hand at the *end* of month i

d_i = *demand* during month i

$$x_i = x_{i-1} + u_i + v_i - d_i; i = 1,2,..,n; \quad x_0 \text{ is known}$$

The LP problem is :

$$\min_{x_i, u_i, v_i} \sum_{i=1}^{n} sx_i + bu_i + cv_i$$

$$\text{s.t.} \quad x_i \geq 0; \quad 0 \leq u_i \leq r; \quad v_i \geq 0; \quad i = 1,2,..,n$$

- Problem 5: Exercise 1.4

Let $|x_2 - 10| = u; \quad |x_1 + 2| = v; \quad |x_2| = w$

$$\min_{x_1, x_2, u \ge 0, v \ge 0, w \ge 0} 2x_1 + 3u$$

$$\begin{aligned} s.t. \quad & v + w \le 5 \\ & x_2 - 10 \le u \quad \text{and} \quad -x_2 + 10 \le u \\ & x_1 + 2 \le v \quad \text{and} \quad -x_1 - 2 \le v \\ & x_2 \le w \quad \text{and} \quad -x_2 \le w \end{aligned}$$

$solution: x_1 = -2, x_2 = 5, u = 5, v = 0, w = 5$

optimal cost $= 11$.

- Problem 6: Exercise 1.8

$$\min \max_i |I_i - I_i^*| = \min \ z$$

$$\begin{aligned} s.t. \quad & I_i = \sum_{j=1}^{m} a_{ij} p_j \\ & z \ge I_i - I_i^*; \ i = 1,2,..,n \\ & z \ge I_i^* - I_i; \ i = 1,2,..,n \\ & p_j \ge 0; \ j = 1,2,..,m \end{aligned}$$

Problem 7: Exercise 1.13

Method 1: Let γ be an arbitrary scalar and consider

$$\max_{\underline{x}} \ \underline{f}^T \underline{x} + g$$

$$\text{subject to } A\underline{x} \le \underline{b}$$

$$\underline{c}^T \underline{x} + d \le \gamma(\underline{f}^T \underline{x} + g)$$

If the optimal cost is positive, it means that there exists some $\underline{x}$ which is a feasible solution of the linear fractional programming problem and optimal cost is less than γ. Initially, check if solution is less than $(K+L)/2$. If it is, solution must be in $[K,(K+L)/2]$. Otherwise, the solution must be in $[(K+L)/2, L]$. In any case, the interval is reduced to $(L-K)/2$. Continuing this way, after k iterations, the cost is reduced to an interval of length $(L-K)/2^k$.

Method 2: Method of Charnes and Cooper

Let $z = 1/\ \underline{f}^T \underline{x} + g$ and $\underline{y} = z\underline{x}$

$$\min_{\underline{y} \ge \underline{0}, z \ge 0} \ \underline{c}^T \underline{y} + dz$$

$$\text{subject to } A\underline{y} - \underline{b}z \le \underline{0}$$

$$\underline{f}^T \underline{y} + gz = 1$$

Problem 8: Exercise 1.15

$a)$ Decision variables:

x_1 = number of products of the first type produced

x_2 = number of products of the second type produced

$$\max_{x_1 \geq 0, x_2 \geq 0} 7.8x_1 + 7.1x_2$$

$$\text{subject to } \frac{1}{4}x_1 + \frac{1}{3}x_2 \leq 90$$

$$\frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80$$

$$\Rightarrow x_1 = 360, x_2 = 0$$

$$Profit = \$2808.$$

$b.i)$ Let x_3 = number of hours of overtime assembly labor

$$\max_{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0} 7.8x_1 + 7.1x_2 - 7x_3$$

$$\text{subject to } \frac{1}{4}x_1 + \frac{1}{3}x_2 - x_3 \leq 90$$

$$\frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80$$

$$x_3 \leq 50$$

$$\Rightarrow x_1 = 560, x_2 = 0, x_3 = 50$$

$$Profit = \$4018.$$

$b.ii)$ Solve two problems:

$$\max_{x_1 \geq 0, x_2 \geq 0} 7.8x_1 + 7.1x_2$$

$$\text{subject to } \frac{1}{4}x_1 + \frac{1}{3}x_2 \leq 90$$

$$\frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80$$

$$1.2x_1 + 0.9x_2 \leq 300$$

$$\Rightarrow x_1 = 108.6, x_2 = 188.6$$

$$Profit = \$2185.7$$

$$\max_{x_1 \geq 0, x_2 \geq 0} 7.92x_1 + 7.19x_2$$

$$\text{subject to } \frac{1}{4}x_1 + \frac{1}{3}x_2 \leq 90$$

$$\frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80$$

$$1.2x_1 + 0.9x_2 \geq 300$$

$$\Rightarrow x_1 = 360, x_2 = 0$$

$$Profit = \$2851.$$

Select second solution because it gives higher profit

- Problem 9

$$\min_{\underline{w}, w_0} \sum_{n=1}^{N} \max[0., 1 - t_n(\underline{w}^T \underline{\phi}(\underline{x}_n) + w_0)] + \lambda \sum_{k=1}^{n_w} |w_k|$$

Let $z_n = \max[0., 1 - t_n(\underline{w}^T \underline{\phi}(\underline{x}_n) + w_0)]; n = 1, 2, .., N$

$$y_k = |w_k|$$

Then the LP problem is:

$$\min_{\underline{w}, w_0, \underline{z}, \underline{y}} \sum_{n=1}^{N} z_n + \lambda \sum_{k=1}^{n_w} y_k$$

$$\begin{aligned}
s.t.\quad & z_n + t_n(\underline{w}^T \underline{\phi}(\underline{x}_n) + w_0) \geq 1; n = 1, 2, .., N \\
& z_n \geq 0; n = 1, 2, .., N \\
& w_k \leq y_k; k = 1, 2, .., n_w \\
& -w_k \leq y_k; k = 1, 2, .., n_w \\
& y_k \geq 0; k = 1, 2, .., n_w
\end{aligned}$$