HW set # 3Solutions

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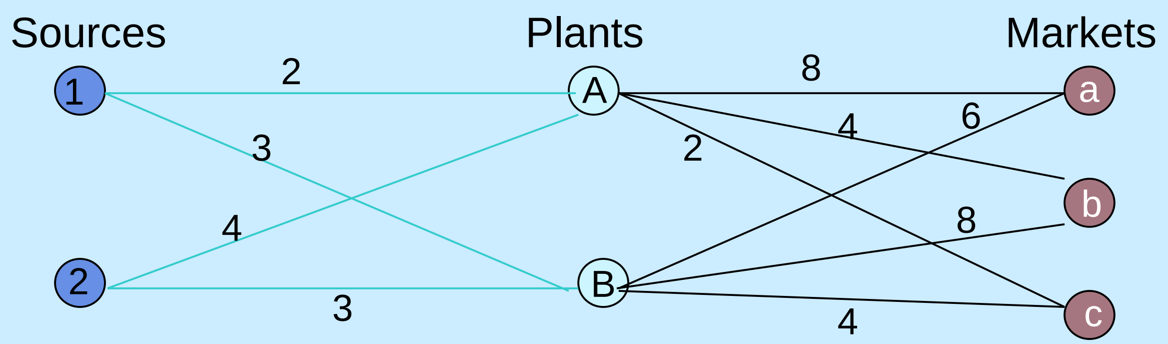
ECE 6108
Linear Programming and Network Flows

Problem 1



Let x_{ij} = source − plant shipping pattern

y_{jk} = plant - market shipping pattern

a) $\min\ (2x_{1A} + 3x_{1B} + 4x_{2A} + 3x_{2B}) +$

$(8y_{Aa} + 4y_{Ab} + 2y_{Ac} + 6y_{Ba} + 8y_{Bb} + 4y_{Bc})$

s.t. $x_{1A} + x_{1B} \le 10;\ x_{2A} + x_{2B} \le 15 \Leftarrow$ source constraints

$y_{Aa} + y_{Ba} \ge 8;\ y_{Ab} + y_{Bb} \ge 14;\ y_{Ac} + y_{Bc} \ge 3 \Leftarrow$ demand constraints

$x_{1A} + x_{2A} = y_{Aa} + y_{Ab} + y_{Ac} \Leftarrow$ conservation constraint at A

$x_{1B} + x_{2B} = y_{Ba} + y_{Bb} + y_{Bc} \Leftarrow$ conservation constraint at B

Network Flow Problem

Solution:

$x_{1A} = 10;\ x_{2A} = 7;\ x_{2B} = 8$

$y_{Ab} = 14;\ y_{Ac} = 3;\ y_{Ba} = 8$

cost = \$182

Problem 1: part b

(b) since plants have unlimited capacity, need find shortest paths only

$1 \to a \Rightarrow \min(10,9) = 9 \Rightarrow \text{path } 1 \to B \to a$

$1 \to b \Rightarrow \min(6,11) = 6 \Rightarrow \text{path } 1 \to A \to b$

$1 \to c \Rightarrow \min(4,7) = 4 \Rightarrow \text{path } 1 \to A \to c$

$2 \to a \Rightarrow \min(12,9) = 9 \Rightarrow \text{path } 2 \to B \to a$

$2 \to b \Rightarrow \min(8,11) = 8 \Rightarrow \text{path } 2 \to A \to b$

$2 \to c \Rightarrow \min(6,7) = 6 \Rightarrow \text{path } 2 \to A \to c$

$$\min\ (9z_{1a} + 6z_{1b} + 4z_{1c} + 9z_{2a} + 8z_{2b} + 6z_{2c})$$

$$\text{s.t.}\quad z_{1a} + z_{1b} + z_{1c} \le 10\,;\ z_{2a} + z_{2b} + z_{2c} \le 15$$

$$z_{1a} + z_{2a} \ge 8;\ z_{1b} + z_{2b} \ge 14;\ z_{1c} + z_{2c} \ge 3$$

$$z_{1a}, z_{1b}, z_{1c}, z_{2a}, z_{2b}, z_{2c} \ge 0$$

Transportation Problem

Solution:

$z_{1b} = 10$

$z_{2a} = 8;\ z_{2b} = 4;\ z_{2c} = 3$

$\cos t = \$182.$

- Problem 1: part c

(c) The two transportation problems are:

$$\min\ 2x_{1A}+3x_{1B}+4x_{2A}+3x_{2B}$$

$$\begin{aligned} \text{s.t.}\quad & x_{1A}+x_{1B}\le 10\\ & x_{2A}+x_{2B}\le 15\\ & x_{1A}+x_{2A}\ge 8\\ & x_{2A}+x_{2B}\ge 17\\ & x_{1A},x_{1B},x_{2A},x_{2B}\ge 0 \end{aligned}$$

Solution:
$x_{1A}=8;\ x_{1B}=2$
$x_{2B}=15$
$\text{cost}=\$67.$

$$\min\ 8y_{Aa}+4y_{Ab}+2y_{Ac}+6y_{Ba}+8y_{Bb}+4y_{Bc}$$

$$\begin{aligned} \text{s.t.}\quad & y_{Aa}+y_{Ab}+y_{Ac}\le 8\\ & y_{Ba}+y_{Bb}+y_{Bc}\le 17\\ & y_{Aa}+y_{Ba}\ge 8\\ & y_{Ab}+y_{Bb}\ge 14\\ & y_{Ac}+y_{Bc}\ge 3\\ & y_{Aa},y_{Ab},y_{Ac}\ge 0\\ & y_{Ba},y_{Bb},y_{Bc}\ge 0 \end{aligned}$$

Solution:
$y_{Ab}=8$
$y_{Ba}=8,\ y_{Bb}=6,\ y_{Bc}=3$
$\text{cost}=\$140.$

Total cost = $207.

- Problem 2:

Let x_1 =# of parts of type 1 produced per day

x_2 =# of parts of type 2 produced per day

Assemblies = $\min(x_1, x_2)$

Time on drilling machine: $3x_1 + 5x_2 \leq 480$

Time on *each* milling machine: $4x_1 + 3x_2 \leq 480$

Also, need: $|3x_1 + 5x_2 - (4x_1 + 3x_2)| \leq 30$

The problem is:

$$\max z$$

$$\text{s.t. } z - x_1 \leq 0;\ z - x_2 \leq 0$$

$$-x_1 + 2x_2 \leq 30;\ x_1 - 2x_2 \leq 30$$

$$3x_1 + 5x_2 \leq 480$$

$$4x_1 + 3x_2 \leq 480$$

$$z, x_1, x_2 \geq 0$$

Solution:

$z = 51.8 \Rightarrow 51$

$x_1 = 73.6;\ x_2 = 51.8$

- Problem 3:

$$\min -2x_1 - 4x_2 - x_3 - x_4$$

$$\text{s.t.} \quad x_1 + 3x_2 + x_4 \le 4$$

$$2x_1 + x_3 \le 3$$

$$x_2 + 4x_3 + x_4 \le 3$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Solution:
$x_1 = 1.24; x_2 = 0.92$
$x_3 = 0.52; x_4 = 0$
$\text{cost} = -6.68$
$\Rightarrow \text{Profit} = 6.68$

$a)$ simplex multipliers : $\underline{\lambda}^T = \underline{c}_B^T B^{-1}$

$$= [-2\ -4\ -1]\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix}^{-1} = [-2\ -4\ -1]\begin{bmatrix} 0.04 & 0.48 & -0.12 \\ 0.32 & -0.16 & 0.04 \\ -0.08 & 0.04 & 0.24 \end{bmatrix} = [-1.28\ -0.36\ -0.16]$$

Reduced cost vector : $\underline{p} = \underline{c} - A^T \underline{\lambda} = [0\ 0\ 0\ 0.44]$

- Problem 3:

$$f = \underline{c}_B^T \underline{x}_B + \underline{p}^T \underline{x}_N = f_0 + \underline{p}^T \underline{x}_N \geq f^*$$

$$f^* - f_0 \leq \underline{p}^T \underline{x}_N$$

$$\Rightarrow | f^* - f_0 | \leq | \max_j (p_j) | \, \| \underline{x}_N \|_1 \leq | \max_j (p_j) | \, s = Ms$$

$$\Rightarrow | f^* - f_0 | \leq Ms \leq \varepsilon \Rightarrow f_0 - f^* \leq \varepsilon \text{ since } f_0 \geq f^*$$

- Problem 4:

Know $\underline{y}_k = B^{-1} \underline{a}_k$. Reduction in cost $= p_k x_{Nk}$

Let current $\underline{x}_B = \underline{\beta}$

To maximize $p_k x_{Nk}$, clearly select k such that $\max_{k: y_{ik} > 0} \left(\frac{-p_k \beta_i}{y_{ik}} \right)$

Problem 5:

Let r_i denote refrigerator production in quarter $i, i=1,2,3,4$

s_i denote stove production in quarter $i, i=1,2,3,4$

w_i denote dish washer production in quarter $i, i=1,2,3,4$

The LP formulation is:

$$\begin{aligned}\min\ & 5(r_1-1500)+5(r_1+r_2-2500)+5(r_1+r_2+r_3-4500)+(r_1+r_2+r_3-5700)\\ & +5(s_1-1500)+5(s_1+s_2-3000)+5(s_1+s_2+s_3-4200)+(s_1+s_2+s_3+s_4-5700)\\ & +5(w_1-1000)+5(w_1+w_2-3000)+5(w_1+w_2+w_3-4500)+(w_1+w_2+w_3+w_4-7000)\end{aligned}$$

$$\begin{aligned}\text{subject to: } & r_1 \ge 1650 \quad r_1+r_2 \ge 2650 \quad r_1+r_2+r_3 \ge 5850\\ & s_1 \ge 1650 \quad s_1+s_2 \ge 3150 \quad s_1+s_2+s_3 \ge 4350 \quad s_1+s_2+s_3+s_4 \ge 5850\\ & w_1 \ge 1150 \quad w_1+w_2 \ge 3150 \quad w_1+w_2+w_3 \ge 4650 \quad w_1+w_2+w_3+w_4 \ge 7150\\ & 2r_1+4s_1+3w_1 \le 18000 \quad 2r_2+4s_2+3w_2 \le 18000 \quad 2r_3+4s_3+3w_3 \le 18000 \quad 4s_4+3w_4 \le 18000\end{aligned}$$

	Q1	Q2	Q3	Q4
Refrigerators	1650	1000	3200	0
Stoves	1650	1500	1200	1500
Dish Washers	1150	2000	1500	2500

Inventory cost=$15,000.

- Problem 6:

$$\min \ -4x_1 - x_2 - 3x_3 - 2x_4$$

$$\begin{aligned} \text{s.t.} \quad 2x_1 + 2x_2 + x_3 + 2x_4 &\le 6 \\ x_2 + 2x_3 + 3x_4 &\le 4 \\ 2x_1 + x_2 \qquad\qquad &\le 5 \\ x_2 \qquad\qquad &\le 1 \\ -x_3 + 2x_4 &\le 2 \\ x_3 + 2x_4 &\le 6 \\ x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Need to use Bland's rule to avoid cycling

Solution :

$x_1 = 2 \ \ x_2 = 0$

$x_3 = 2 \ \ x_4 = 0$

$\text{cost} = -14$

$\underline{\lambda} = [-2 \ -0.5]^T$

$$\text{Iteration } 0: B = I \ , \ \underline{x}_1 = \underline{0}, \underline{x}_2 = \underline{0}, \underline{\lambda}^T = [0\ 0\ 0\ 0]$$

$$\text{bfs} = [s_1 \ s_2 \ \delta_1 \ \mu_1]^T = [6\ 4\ 1\ 1]^T$$

Subproblems :

$$\begin{aligned} &\min \ -4x_1 - x_2 \\ &s.t. \quad 2x_1 + x_2 \le 5 \\ &\qquad\qquad x_2 \le 1 \\ &opt.\text{sol.} = (2.5\ 0) \\ &\text{cost}: p_1 = -10 \end{aligned}$$

$$\begin{aligned} &\min \ -3x_3 - 2x_4 \\ &s.t. \quad -x_3 + 2x_4 \le 2 \\ &\qquad\quad x_3 + 2x_4 \le 6 \\ &opt.\text{sol.} = (6\ 0) \\ &\text{cost}: p_2 = -18 \end{aligned}$$

Bring δ_2 into basis

$$\begin{bmatrix} L_1 \underline{x}_1^{(1)} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2.5 \\ 0 \end{bmatrix} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \underline{a}_k$$

- Problem 6:

$$B^{-1}\underline{a}_k = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \frac{\beta_i}{\alpha_{ik}} = \begin{bmatrix} 6/5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \text{column 3 must go}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \underline{x}_B = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \delta_2 \\ \mu_1 \end{bmatrix}; \underline{\lambda}^T = [0\ 0\ \text{-}10\ 0]^T$$

Iteration 1: Subproblem solutions do not change since $\underline{\lambda}_0 = \underline{0}$ (first two comp.)

Bring μ_2 into basis

$$\begin{bmatrix} L_2 \underline{x}_2^{(2)} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \end{bmatrix} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 0 \\ 1 \end{bmatrix} = \underline{a}_k$$

$$B^{-1}\underline{a}_k = \begin{bmatrix} 6 \\ 12 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \frac{\beta_i}{\alpha_{ik}} = \begin{bmatrix} 1/6 \\ 1/3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{column 1 must go}$$

$$B^{-1} = \begin{bmatrix} 1/6 & 0 & -5/6 & 0 \\ -2 & 1 & 10 & 0 \\ 0 & 0 & 1 & 0 \\ -1/6 & 0 & 5/6 & 1 \end{bmatrix}; \quad \underline{x}_B = \begin{bmatrix} 1/6 \\ 2 \\ 1 \\ 5/6 \end{bmatrix} = \begin{bmatrix} \mu_2 \\ s_2 \\ \delta_2 \\ \mu_1 \end{bmatrix}; \underline{\lambda}^T = [\text{-}3\ 0\ 5\ 0]$$

Problem 6:

Iteration 2 : subproblems

$$\min\ 2x_1 + 5x_2 - 5$$
$$s.t.\quad 2x_1 + x_2 \le 5$$
$$x_2 \le 1$$
$$opt.\ \text{sol.} = (0\ 0)$$
$$\text{cost}: p_1 = -5$$

$$\min\ 4x_4$$
$$s.t.\quad -x_3 + 2x_4 \le 2$$
$$x_3 + 2x_4 \le 6$$
$$opt.\ \text{sol.} = (0\ 0)$$
$$\text{cost}: p_2 = 0$$

Bring δ_3 into basis

$$\begin{bmatrix} L_1 \underline{x}_1^{(3)} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \underline{a}_k$$

$$B^{-1}\underline{a}_k = \begin{bmatrix} -5/6 \\ 10 \\ 1 \\ 5/6 \end{bmatrix} \Rightarrow \frac{\beta_i}{\alpha_{ik}} = \begin{bmatrix} * \\ 1/5 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{column 2 must go}$$

$$B^{-1} = \begin{bmatrix} 0 & 5/60 & 0 & 0 \\ -1/5 & 1/10 & 1 & 0 \\ 1/5 & -1/10 & 0 & 0 \\ 0 & -5/60 & 0 & 1 \end{bmatrix};\quad \underline{x}_B = \begin{bmatrix} 1/3 \\ 1/5 \\ 4/5 \\ 2/3 \end{bmatrix} = \begin{bmatrix} \mu_2 \\ \delta_3 \\ \delta_2 \\ \mu_1 \end{bmatrix};\ \underline{\lambda}^T = [\text{-}2\ \text{-}0.5\ 0\ 0]$$

- Problem 6:

Iteration 3: subproblems

$\Rightarrow$ optimal

$$\min 3.5x_1 \quad s.t. \quad 2x_1 + x_2 \le 5, \quad x_2 \le 1$$

$opt.\,\text{sol.} = (0\ 0)$

$\text{cost}: p_1 = 0$

$$\min 3.5x_4 \quad s.t. \quad -x_3 + 2x_4 \le 2, \quad x_3 + 2x_4 \le 6$$

$opt.\,\text{sol.} = (0\ 0)$

$\text{cost}: p_2 = 0$

$$\underline{x}^* = 1/3\begin{bmatrix}0\\0\\6\\0\end{bmatrix} + 1/5\begin{bmatrix}0\\0\\0\\0\end{bmatrix} + 4/5\begin{bmatrix}2.5\\0\\6\\0\end{bmatrix} + 2/3\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\2\\0\end{bmatrix}$$

- Problem 7:

$$\min \underline{c}^T \underline{x} \text{ s.t } A\underline{x} = \underline{b}, \underline{x} \ge \underline{0}$$

$$a)\ c_i^{'} = c_i - r\underline{\lambda}^T \underline{a}_i;\ r = H / \underline{c}_B^T \underline{x}_B;\ \underline{\lambda}^T = \underline{c}_B^T B^{-1};\ \underline{x}_B = B^{-1}\underline{b}$$

$f^* = \underline{c}_B^T B^{-1}\underline{b}$. If $\underline{x}_B$ is still optimal with new $\underline{c}$ vector, then

$$f_{new}^* = \underline{c}_B^T B^{-1}\underline{b} - \frac{H}{f^*}\underline{\lambda}^T B B^{-1}\underline{b} = f^*(1-r) = \underline{\lambda}^T(1-r)\underline{b}$$

$$\Rightarrow \underline{\lambda}_{new} = \underline{\lambda}(1-r)$$

To show that $\underline{x}_B$ is still optimal, consider the reduced cost vector

- Problem 7:

$$p_i' = c_i' - \lambda_{new}^T \underline{a}_i = c_i - r\underline{\lambda}^T \underline{a}_i - (1-r)\underline{\lambda}^T \underline{a}_i \geq 0 \text{ for non-basic } i$$

$$p_i' = 0 \text{ for basic } i \Rightarrow \underline{x}_B \text{ is still optimal}$$

b) $c_i' = c_i - \underline{r}^T \underline{a}_i;\ \underline{r}^T = [r_1\, r_2 \ldots\ldots r_m]; r_i = H_i / b_i$

To show that $\underline{x}_B$ is still optimal, consider

$$\lambda_{new}^T = \underline{c}_B^{'T} B^{-1} = (\underline{c}_B^T - \underline{r}^T B)B^{-1} = (\underline{\lambda}^T - \underline{r}^T) \Rightarrow \underline{\lambda}_{new} = \underline{\lambda} - \underline{r}$$

$$p_i' = c_i' - \underline{\lambda}_{new}^T \underline{a}_i = c_i - \underline{r}^T \underline{a}_i - (\underline{\lambda}^T - \underline{r}^T)\underline{a}_i \geq 0 \Rightarrow \text{still optimal}$$

$$f_{new}^* = \underline{\lambda}_{new}^T \underline{b} = (\underline{\lambda}^T - \underline{r}^T)\underline{b} = f^* - \underline{r}^T \underline{b} = f^* - \sum_{i=1}^{m} H_i$$

Problem 8

LP Problem is:

$$\min\ (52000 - 44P_1 - 36P_2 - 44P_3 - 40P_4)$$

$$s.t.\quad 2P_1 + P_2 + 2P_3 + 5P_4 = 2000$$

$$P_1 + P_2 + P_3 = 1000$$

$$P_1 \le 500$$

$$P_2 \le 750$$

$$P_3 \le 500$$

$$P_4 \le 750$$

$$P_i \ge 0; i = 1, 2, 3, 4$$

Solution 1:

$P_1 = 0$

$P_2 = 750$ from $10' x 3000'$

$P_3 = 250$ from $11' x 2000'$

$P_4 = 150$ from $11' x 2000'$

wastage = 8000 sq. ft.

Solution 2:

$P_1 = 500$

$P_2 = 500$ from $10' x 3000'$

$P_3 = 0$ from $11' x 2000'$

$P_4 = 100$ from $11' x 2000'$

wastage = 8000 sq. ft.

Solution 3: Same as solution 2 with P_1 & P_3 interchanged.

Soluton 4: Same as solution 1 with P_1 & P_3 interchanged.

- Problem 9: Exercise 3.22

a) If $b=0$, then $\underline{x}=\underline{0}$ is feasible.

If $b>0$, then the problem is feasible if at least one $a_j>0$.

If $b<0$, then the problem is feasible if at least one $a_j<0$.

b) assume wlog that $b>0$. Let k be the index such that

$$k=\arg\min_j c_j\frac{b}{a_j} \text{ and } x_k=\frac{b}{a_k}$$

$dual: \max \lambda b$

$s.t. a_i\lambda \le c_i, i=1,2,..,n$

$\Rightarrow \max_{a_i<0}\frac{c_i}{a_i}\le\lambda\le\min_{a_i>0}\frac{c_i}{a_i}$

$b>0\Rightarrow\lambda=\min_{i:a_i>0}\left(\frac{c_i}{a_i}\right)$

$b<0\Rightarrow\lambda=\max_{i:a_i<0}\left(\frac{c_i}{a_i}\right)$

$$so, x_i=\begin{cases}\min_{i:a_i>0}\left(\frac{b}{a_i}\right); b>0\\ \max_{i:a_i<0}\left(\frac{b}{a_i}\right); b<0\end{cases}$$

- Problem 10: Exercise 6.3

a) Let $\underline{x}=\delta_1\underline{x}^1+\delta_2\underline{x}^2=[20\delta_1,10\delta_2,10\delta_2,20\delta_2,10\delta_1,10\delta_1]$

The restricted master problem is:

$$\max\ 20\delta_1+10\delta_2 \text{ s.t. } 30\delta_1\le 15, \delta_1+\delta_2=1, \delta_1,\delta_2\ge 0$$

The solution $\delta_1=\delta_2=1/2$. optimal value is 15. $\lambda_1=1/3, \lambda_2=10$

$$\underline{\lambda}^T=[20\quad 10]\begin{bmatrix}30 & 0\\ 1 & 1\end{bmatrix}^{-1}=[20\quad 10]\begin{bmatrix}1/30 & 0\\ -1 & 1\end{bmatrix}=[1/3\quad 10]$$

b) $\max\ x_{12}+x_{22}+x_{23}-1/3x_{11}-1/3x_{23}$

solution is: $(x_{11}=10, x_{12}=10, x_{13}=0, x_{21}=10, x_{22}=0, x_{23}=10)$

and the optimal value is $z=13.33$

c) reduced cost $=$ z-$\lambda_2=3.333$

d) Upper bound is: $15+3.33=18.33$. The actual value is 17.5.

Recall maximization.
$\Rightarrow$ Dual feasible cost ≥ primal