HW set # 4 solutions

Prof. Krishna R. Pattipati

Dept. of Electrical and Computer Engineering
University of Connecticut
Contact: krishna@engr.uconn.edu (860) 486-2890

ECE 6108
Linear Programming and Network Flows

- Problem 1:

$$\min \; -2x_1 - 4x_2 - x_3 - x_4$$

$$\text{s.t.} \quad x_1 + 3x_2 + x_4 \le 4$$

$$2x_1 + x_3 \le 3$$

$$x_2 + 4x_3 + x_4 \le 3$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Solution:

$x_1 = 1.24; x_2 = 0.92$

$x_3 = 0.52; x_4 = 0$

$\text{cost} = -6.68$

$\Rightarrow \text{Profit} = 6.68$

$a)$ simplex multipliers : $\underline{\lambda}^T = \underline{c}_B^T B^{-1}$

$$= [-2\;-4\;-1]\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix}^{-1} = [-2\;-4\;-1]\begin{bmatrix} 0.04 & 0.48 & -0.12 \\ 0.32 & -0.16 & 0.04 \\ -0.08 & 0.04 & 0.24 \end{bmatrix} = [-1.28\; -0.36\; -0.16]$$

Reduced cost vector : $\underline{p} = \underline{c} - A^T \underline{\lambda} = [0\;0\;0\;0.44]$

Problem 1:

$$B^{-1} = \begin{bmatrix} 0.04 & 0.48 & -0.12 \\ 0.32 & -0.16 & 0.04 \\ -0.08 & 0.04 & 0.24 \end{bmatrix}$$

$\underline{y}_1 \quad \underline{y}_2 \quad \underline{y}_3$

To find range of b_i, compute column $\underline{y} = (B^{-1})_i$

$$\max_{\{y_j : y_j > 0\}} \left(-\frac{x_{Bj}}{y_j}\right) \le \delta_i \le \min_{\{y_j : y_j < 0\}} \left(-\frac{x_{Bj}}{y_j}\right)$$

$$\max(-1.24/0.04, -0.92/0.32) \le \delta_1 \le 0.52/0.08$$

$$\Rightarrow -2.875 \le \delta_1 \le 6.5 \Rightarrow 4 - 2.875 \le 4 + \delta_1 \le 10.5$$

$$\Rightarrow 1.125 \le b_1 \le 10.5$$

$$\max(-1.24/0.48, -0.52/0.04) \le \delta_2 \le 0.92/0.16$$

$$\Rightarrow -2.5833 \le \delta_2 \le 5.75 \Rightarrow 3 - 2.5833 \le 3 + \delta_2 \le 8.75$$

$$\Rightarrow 0.4167 \le b_2 \le 8.75$$

$$\max(-0.92/0.04, -0.52/0.24) \le \delta_3 \le 1.24/0.12$$

$$\Rightarrow -2.1667 \le \delta_3 \le 10.33 \Rightarrow 3 - 2.1667 \le 3 + \delta_3 \le 13.33$$

$$\Rightarrow 0.8333 \le b_3 \le 13.33$$

Solution:

$x_1 = 1.24; x_2 = 0.92$

$x_3 = 0.52; x_4 = 0$

Problem 1:

$$B^{-1} = \begin{bmatrix} 0.04 & 0.48 & -0.12 \\ 0.32 & -0.16 & 0.04 \\ -0.08 & 0.04 & 0.24 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

b) Reduced cost vector: $\underline{p}^T = [0\ 0\ 0\ 0.44\ 1.28\ 0.36\ 0.16]$

The cost coefficient c_4 can be reduced by $0.44 \Rightarrow \Delta c_4 \geq -0.44 \Rightarrow c_4 \geq -1.44 \Rightarrow orig - c_4 \leq 1.44$

To find allowable changes in c_i for basic variables, compute $BIN = (B^{-1})N$

$$BIN = \begin{bmatrix} -0.08 & 0.04 & 0.48 & -0.12 \\ 0.36 & 0.32 & -0.16 & 0.04 \\ 0.16 & -0.08 & 0.04 & 0.24 \end{bmatrix}$$

Use $[\underline{c}_B^T + \delta_i \underline{e}_i^T]B^{-1}\underline{a}_j \leq c_j \forall$nonbasic j

$\Rightarrow \delta_i \underline{e}_i^T B^{-1} \underline{a}_j \leq c_j - \underline{\lambda}^T \underline{a}_j = p_j$

$$\max_{i \in NBV:(B^{-1}\underline{a}_i)_l < 0} \left(\frac{p_i}{(B^{-1}\underline{a}_i)_l} \right) \leq \delta_j \leq \min_{i \in NBV:(B^{-1}\underline{a}_i)_l > 0} \left(\frac{p_i}{(B^{-1}\underline{a}_i)_l} \right)$$

$$\max(-\frac{0.44}{0.08}, -\frac{0.16}{0.12}) \leq \delta_1 \leq \min(\frac{1.28}{0.04}, \frac{0.36}{0.48}) \Rightarrow -1.33 \leq \delta_1 \leq 0.75 \Rightarrow -3.33 \leq c_1 \leq -1.25 \Rightarrow 1.25 \leq orig - c_1 \leq 3.33$$

$$-\frac{0.36}{0.16} \leq \delta_2 \leq \min(\frac{0.44}{0.36}, \frac{1.28}{0.32}, \frac{0.16}{0.04}) \Rightarrow -2.25 \leq \delta_2 \leq 1.222 \Rightarrow -6.25 \leq c_2 \leq -2.778 \Rightarrow 2.778 \leq orig - c_2 \leq 6.25$$

$$-\frac{1.28}{0.08} \leq \delta_3 \leq \min(\frac{0.44}{0.16}, \frac{0.36}{0.04}, \frac{0.16}{0.24}) \Rightarrow -16 \leq \delta_3 \leq 0.67 \Rightarrow -17 \leq c_3 \leq -0.33 \Rightarrow 0.33 \leq orig - c_3 \leq 17$$

- Problem 1:

c) simplex multiplier vector: $\underline{\lambda}^T = [-1.28\ \text{-}0.36\ \text{-}0.16]$

$$\Delta f = \underline{\lambda}^T \Delta \underline{b}$$
$$= -1.28\Delta b_1 - 0.36\Delta b_2 - 0.16\Delta b_3$$

$$d)\ \Delta f = \underline{x}_B^T \Delta \underline{c}$$
$$= 1.24\Delta c_1 + 0.92\Delta c_2 + 0.52\Delta c_3$$

Problem 2

primal :

x_i = Number of processes of type i used, $i = 1, 2, 3$

$$\max_{x_i \geq 0} f = 38(4x_1 + x_2 + 3x_3) + 33(3x_1 + x_2 + 4x_3) - 51x_1 - 11x_2 - 40x_3 = 200x_1 + 60x_2 + 206x_3$$

$$\Rightarrow \min_{x_i \geq 0} -200x_1 - 60x_2 - 206x_3$$

$$\text{s.t.} \quad 3x_1 + x_2 + 5x_3 \leq 8 \Rightarrow 3x_1 + x_2 + 5x_3 + s_1 = 8$$

$$5x_1 + x_2 + 3x_3 \leq 5 \Rightarrow 5x_1 + x_2 + 3x_3 + s_2 = 5$$

Iteration 1:

$$B = I; \underline{x}_B = \begin{bmatrix} 8 \\ 5 \end{bmatrix}; \underline{\lambda}^T = \begin{bmatrix} 0 & 0 \end{bmatrix}; \underline{p}^T = [-200 - 60 - 206\ 0\ 0]$$

Bring x_3 into basis. $\underline{\alpha} = B^{-1}\underline{a}_3 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}; \theta = \min\{\frac{8}{5}, \frac{5}{3}\} = \frac{8}{5} \Rightarrow s_1$ should go out.

Iteration 2:

$$B = \begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix}; B^{-1} = \begin{bmatrix} 1/5 & 0 \\ -3/5 & 1 \end{bmatrix}; \underline{x}_B = \begin{bmatrix} 8/5 \\ 1/5 \end{bmatrix}; \underline{\lambda}^T = \begin{bmatrix} -206/5 & 0 \end{bmatrix}; \underline{p}^T = [-382/5\ -94/5\ \ 0\ \ 206/5\ \ 0]$$

Bring x_1 into basis. $\underline{\alpha} = B^{-1}\underline{a}_1 = \begin{bmatrix} 3/5 \\ 16/5 \end{bmatrix}; \theta = \min\{\frac{8}{3}, \frac{1}{16}\} = \frac{1}{16} \Rightarrow s_2$ should go out.

Iteration 3:

$$B = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}; B^{-1} = \begin{bmatrix} 5/16 & -3/16 \\ -3/16 & 5/16 \end{bmatrix}; \underline{x}_B = \begin{bmatrix} 25/16 \\ 1/16 \end{bmatrix}; \underline{\lambda}^T = \begin{bmatrix} -215/8 & -191/8 \end{bmatrix}; \underline{p}^T = [0\ -37/4\ 0\ 215/8\ 191/8]$$

Bring x_2 into basis. $\underline{\alpha} = B^{-1}\underline{a}_2 = \begin{bmatrix} 1/8 \\ 1/8 \end{bmatrix}; \theta = \min\{\frac{25}{2}, \frac{1}{2}\} = \frac{1}{2} \Rightarrow x_1$ should go out.

Problem 2

Iteration 3:

$$B=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}; B^{-1}=\begin{bmatrix}5/16 & -3/16\\ -3/16 & 5/16\end{bmatrix}; \underline{x}_B=\begin{bmatrix}25/16\\ 1/16\end{bmatrix}; \underline{\lambda}^T=[-215/8 \quad -191/8]; \underline{p}^T=[0 \; -37/4 \; 0 \; 215/8 \; 191/8]$$

Bring x_2 into basis. $\underline{\alpha}=B^{-1}\underline{a}_2=\begin{bmatrix}1/8\\ 1/8\end{bmatrix}; \theta=\min\{\frac{25}{2},\frac{1}{2}\}=\frac{1}{2}\Rightarrow x_1$ should go out.

Iteration 4:

$$B=\begin{bmatrix}5 & 1\\ 3 & 1\end{bmatrix}; B^{-1}=\begin{bmatrix}1/2 & -1/2\\ -3/2 & 5/2\end{bmatrix}; \underline{x}_B=\begin{bmatrix}3/2\\ 1/2\end{bmatrix}; \underline{\lambda}^T=[-13 \quad -47]; \underline{p}^T=[74 \; 0 \; 0 \; 13 \; 47]$$

$Optimal.\cos t=-339M \Rightarrow Net\ revenue=\$339M$

$b) c_1 \geq -274 \Rightarrow$ in terms of original problem $-c_1 \leq 274 \Rightarrow$ *can increase by* 74

$$B^{-1}N=\begin{bmatrix}-1 & 1/2 & -1/2\\ 8 & -3/2 & 5/2\end{bmatrix}$$

$\Rightarrow -206+\max(-74,-94)\leq c_3 \leq -206+26 \Rightarrow 180 \leq -c_3 \leq 280$

$-60-26/3 \leq c_2 \leq -60+\min(37/4, 94/5) \Rightarrow 203/4 \leq -c_2 \leq 206/3$

For gasoline price rise by p, the new coefficients are $(200+4p, 60+p, 206+3p)$

So, need $4p \leq 74; p \leq 26/3; 3p \leq 74 \Rightarrow p \leq 8.67$

$c)\ \lambda_2=-47 \Rightarrow$ For original problem, dual price=\$47.

Crude A: $b_1: -3 \leq \delta_1 \leq 1$

Crude B: $b_2: -\frac{3}{5} \leq \delta_2 \leq 3 \Rightarrow$ can buy an additional 3M barrels.

Since 47>40, should buy 3M barrels of crude B.

Beyond 3M barrels of oil, dual price becomes \$10. So, 3M barrels is all one should buy.

- Problem 3:

Primal:

$$\max_{x_{ij}\in(0,1)} \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}x_{ij}$$

$$s.t. \quad \sum_{j=1}^{n} x_{ij} = 1 \forall i$$

$$\sum_{i=1}^{n} x_{ij} = 1 \forall j$$

Dual:

$$\min_{\underline{\lambda},\underline{\mu}} \sum_{i=1}^{n}\lambda_i + \sum_{j=1}^{n}\mu_j$$

$$s.t. \quad \lambda_i + \mu_j \geq s_{ij} \forall i, j$$

$$\Rightarrow \min_{\underline{\mu}} \sum_{i=1}^{n} \max_j (s_{ij} - \mu_j) + \sum_{j=1}^{n}\mu_j$$

Lagrangian:

$$L(\{x_{ij}\}, \underline{\lambda}, \underline{\mu}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(s_{ij} - \lambda_i - \mu_j)x_{ij} + \sum_{i=1}^{n}\lambda_i + \sum_{j=1}^{n}\mu_j$$

$$s.t. \quad x_{ij} \in (0,1)$$

Dual: one of minimization

$$q(\underline{\lambda}, \underline{\mu}) = L(\{x^*_{ij}\}, \underline{\lambda}, \underline{\mu}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\max[(s_{ij} - \lambda_i - \mu_j), 0] + \sum_{i=1}^{n}\lambda_i + \sum_{j=1}^{n}\mu_j$$

- Problem 3:

b) If i is allocated to $j \Rightarrow x_{ij} = 1$. From CS conditions: $\lambda_{\mathrm{i}} + \mu_j = s_{ij}$

c) If i is allocated to $j \Rightarrow x_{ij} = 1$ and $x_{ik} = 0 \, \forall k \neq j$ (Recall the constraints)

$\Rightarrow \lambda_{\mathrm{i}} = s_{ij} - \mu_j \geq s_{ik} - \mu_k$.

$\Rightarrow$ activity i is allocated to an individual that provides the greatest profit.

- Problem 4 a:

Primal:

$$\min \underline{c}^T \underline{x}$$
$$\text{s.t. } A\underline{x} \le \underline{b}$$
$$\underline{x} \ge \underline{0}$$

Dual:

$$\max \underline{\lambda}^T \underline{b}$$
$$\text{s.t. } \underline{\lambda}^T A \le \underline{c}^T$$
$$\underline{\lambda} \le \underline{0}$$

Equivalent Dual:

$$\min \underline{\mu}^T \underline{b}$$
$$\text{s.t. } \underline{\mu}^T (-A) \le \underline{c}^T$$
$$\underline{\mu} \ge \underline{0}$$

$$\Rightarrow \underline{b} = \underline{c} \text{ and } A = -A^T \Rightarrow \textit{skew} \text{ symmetric}$$

- Problem 4b:

Primal:

$$\max_{x_i \ge 0} \sum_{i=1}^{n} i x_i$$

$$\text{s.t.} \begin{bmatrix} 1 & 1 & . & 1 \\ 0 & 1 & . & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{x} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{optimal solution}: \underline{x}^* = [0\,0\ldots0\,1]^T$$

Dual:

$$\min_{\lambda_i \ge 0} \sum_{i=1}^{n} \lambda_i$$

$$\text{s.t.} \begin{bmatrix} 1 & 0 & . & 0 \\ 1 & 1 & . & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \underline{\lambda} \ge \begin{bmatrix} 1 \\ 2 \\ . \\ n \end{bmatrix}$$

$$\text{Optimal solution}: \underline{\lambda}^{*T} = [111\ldots..1]^T$$

⇒If you want to maximize the mean of a discrete distribution such that the probabilities satisfy the above triangular inequalities, put all the mass at $x=n$.

- Problem 5:

Primal:

$$\min_{\underline{x} \geq \underline{a}} \underline{c}^T \underline{x}$$

$$A\underline{x} \leq \underline{b}$$

Primal:

$$\min_{\tilde{\underline{x}} \geq \underline{0}} \underline{c}^T (\tilde{\underline{x}} + \underline{a})$$

$$A\tilde{\underline{x}} \leq \underline{b} - A\underline{a}$$

Dual:

$$\max_{\underline{\lambda} \leq \underline{0}} \underline{\lambda}^T (\underline{b} - A\underline{a}) + \underline{c}^T \underline{a}$$

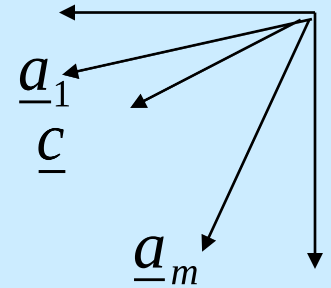
$$\underline{\lambda}^T A \leq \underline{c}^T$$

- Problem 6:

Let A be an mxn matrix.

Farkas' Lemma

$$A\underline{x} \le \underline{0} \Rightarrow \underline{c}^T \underline{x} \le 0 \Leftrightarrow \underline{c}^T = \underline{\lambda}^T A \text{ for some } \underline{\lambda} \ge \underline{0}$$



Primal:	Dual:
$\max\limits_{\underline{x}} \underline{c}^T \underline{x}$	$\min\limits_{\underline{\lambda} \ge \underline{0}} \underline{\lambda}^T \underline{0}$
s.t $A\underline{x} \le \underline{0}$	s.t $\underline{\lambda}^T A = \underline{c}^T$

If the system $A\underline{x} \le \underline{0}$ has at least one solution and if every feasible solution satisfies $\underline{c}^T \underline{x} \le 0$, then the first problem has an optimal solution and the optimal cost is bounded above by 0. By the strong duality theorem, the dual problem also has an optimal solution bounded above by 0 and satisfies the equality constraints. On the other hand, if $\underline{\lambda} \ge \underline{0}$ and $\underline{\lambda}^T A = \underline{c}^T$, and since optimal dual cost $\ge$ optimal primal reward (recall primal maximization), we have $\underline{c}^T \underline{x} \le 0$

- Problem 7:

Consider $\min_{\underline{x}\geq 0} \underline{c}^T \underline{x}$ s.t. $A\underline{x} = \underline{b} \Rightarrow$ Dual : $\max_{\underline{\lambda}} \underline{\lambda}^T \underline{b}$ s.t. $\underline{\lambda}^T A \leq \underline{c}^T$

a) If k^{th} constraint is multiplied by $\mu \neq 0 \Rightarrow$ no change in solution

$$\text{New Dual} : \max_{\underline{w}} \sum_{\substack{j=1\\ j\neq k}}^{m} w_i b_i + w_k b_k \mu$$

$$s.t. \sum_{\substack{j=1\\ j\neq k}}^{m} w_i a_i^T + w_k a_k^T \mu \leq \underline{c}^T \Rightarrow w_i = \lambda_i; i \neq k; w_k = \frac{\lambda_k}{\mu}$$

b) If k^{th} constraint is multiplied by $\mu \neq 0$ and added to r^{th} constraint

$$\text{New Dual} : \max_{\underline{w}} \sum_{\substack{j=1\\ j\neq r}}^{m} w_i b_i + w_r (b_r + b_k \mu)$$

$$s.t. \sum_{\substack{j=1\\ j\neq r}}^{m} w_i \underline{a}_i^T + w_r (\underline{a}_r^T + \underline{a}_k^T \mu) \leq \underline{c}^T$$

$$\Rightarrow w_i = \lambda_i, i \neq k; w_k = \lambda_k + \lambda_r \mu$$

- Problem 7:

b) If μ times the k^{th} row of A is added to $\underline{c}$.

New Dual: $\max_{\underline{w}} \sum_{j=1}^{m} w_i b_i$

$$s.t. \sum_{\substack{i=1 \\ i \neq k}}^{m} w_i \underline{a}_i^T + w_k \underline{a}_k^T \leq \underline{c}^T + \mu \underline{a}_k^T$$

$$\Rightarrow w_i = \lambda_i, i \neq k; w_k = \lambda_k + \mu$$

- Problem 8: (4.7 of Bertismas & Tsitsiklis)

$$\min \max_{i=1,2,..,m} (\underline{a}_i^T \underline{x} - b_i) \Rightarrow \min z \quad \text{s.t. } A\underline{x} - \underline{b} \le z\underline{e}$$

$$a)\ \underline{p}^T A = \underline{0}^T; \underline{p} \ge \underline{0} \text{ and } \underline{p}^T \underline{e} = 1 \Rightarrow \underline{p}^T (A\underline{x} - \underline{b}) = -\underline{p}^T \underline{b} \le z^* = v.$$

b) The dual problem is :

$$\max_{\underline{\lambda} \le \underline{0}} \underline{\lambda}^T \underline{b} \qquad \max_{\underline{p} \ge \underline{0}} - \underline{p}^T \underline{b}$$

$$\text{s.t.} \quad \underline{\lambda}^T A = \underline{0}^T \qquad \underline{p}^T A = \underline{0}^T$$

$$-\underline{\lambda}^T \underline{e} = 1 \qquad \underline{p}^T \underline{e} = 1$$

From the duality theory, the Optimal cost is *v*

- Problem 9: (5.2 of Bertsimas & Tsitsiklis)

  a) Since B is a basis matrix, its determinant is non zero. For small δ,

  (B + δ E) also has non-zero determinant (by Taylor series of the determinant).

  Recall $|B+\delta E| = |B||I+\delta B^{-1}E| \approx |B|[1+\delta tr(B^{-1}E)+O(\delta^2)]$

  b) $\underline{x}_B = (B+\delta E)^{-1}\underline{b} = (I+\delta B^{-1}E)^{-1}B^{-1}\underline{b}$

  $c) B^{-1}\underline{b} > \underline{0} \Rightarrow (B+\delta E)^{-1}\underline{b} > \underline{0}$ for small δ. Also, dual constraints

  $\underline{c}^T - \underline{\lambda}^T \ A \geq \underline{0} \ = \ \underline{c}^T - \underline{c}_B^{\ T} (B+\delta E)^{-1} \ A \geq \underline{0}$ for small δ.

  d) $\underline{c}_B^{\ T}\ \underline{x}_B = \underline{c}_B^{\ T}(B+\delta E)^{-1}\underline{b} = \underline{c}_B^{\ T}(I+\delta B^{-1}E)^{-1}B^{-1}\underline{b}$

  $$\approx \underline{c}_B^{\ T}(I-\delta B^{-1}E)B^{-1}\underline{b} = \underline{c}^T\underline{x}^* - \delta\lambda_1^* x_1^*$$

- Problem 10: (5.4 of Bertsimas & Tsitsiklis)

a) $B(\delta)=\begin{bmatrix}3+\delta & 2\\ 5 & 3\end{bmatrix} \Rightarrow B(\delta)^{-1}\underline{b}=\begin{bmatrix}\dfrac{-2}{3\delta-1}\\ \dfrac{-2+16\delta}{3\delta-1}\end{bmatrix}$

Primal feasibility $\Rightarrow$ $3\delta-1<0$ and $-2+16\delta\le 0 \Rightarrow \delta\le 1/8.$

$b)\ \underline{c}_B^T B(\delta)^{-1}=\begin{bmatrix}-5 & -1\end{bmatrix}\begin{bmatrix}3+\delta & 2\\ 5 & 3\end{bmatrix}^{-1}=\begin{bmatrix}\dfrac{-10}{3\delta-1} & \dfrac{7-\delta}{3\delta-1}\end{bmatrix}$

Reduced costs of x_1, x_2, x_3 and x_4 are:

$$p_1=-5-\frac{5-15\delta}{3\delta-1}=0;\ p_2=-1-\frac{1-3\delta}{3\delta-1}=0;\ p_3=12+\frac{10}{3\delta-1}=\frac{36\delta-2}{3\delta-1}>0\ if\ \delta\le 1/18;$$

$$p_4=-\frac{7-\delta}{3\delta-1}=\frac{\delta-7}{3\delta-1}>0\ if\ \delta<1/3$$

So, $\delta\le 1/18$ for the basis to be optimal!

$c)$ Optimal cost: $\underline{c}_B^T B(\delta)^{-1}\underline{b}=\dfrac{12-16\delta}{3\delta-1}$