HW set # 5 Solution

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ECE 6108
Linear Programming and Network Flows

Problem 1

$$\min \quad -7x_1 + 7x_2 - 2x_3 - x_4 - 6x_5$$
$$s.t. \quad 3x_1 - x_2 + x_3 - 2x_4 = -3$$
$$2x_1 + x_2 + x_4 + x_5 = 4$$
$$-x_1 + 3x_2 - 3x_4 + x_6 = 12$$
$$x_i \geq 0, i = 1, 2, .., 6$$

Dual: $\max \; -3\lambda_1 + 4\lambda_2 + 12\lambda_3$

$$3\lambda_1 + 2\lambda_2 - \lambda_3 \leq -7$$
$$-\lambda_1 + \lambda_2 + 3\lambda_3 \leq 7$$
$$\lambda_1 \leq -2$$
$$-2\lambda_1 + \lambda_2 - 3\lambda_3 \leq -1$$
$$\lambda_2 \leq -6$$
$$\lambda_3 \leq 0$$

Iteration 0: Initial Dual feasible solution: $\lambda_1 = -2; \lambda_2 = -6; \; \lambda_3 = 0; B = I; \underline{x}_B = \begin{bmatrix} -3 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_5 \\ x_6 \end{bmatrix}$

$$x_{B1} < 0 \Rightarrow [row1of \;\; B^{-1}]N = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & -2 \\ 2 & 1 & 1 \\ -1 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 \end{bmatrix}$$

$$\underline{\lambda}^T N = \begin{bmatrix} -2 & -6 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & -2 \\ 2 & 1 & 1 \\ -1 & 3 & -3 \end{bmatrix} = \begin{bmatrix} -18 & -4 & -2 \end{bmatrix}$$

$\varepsilon = \min(\frac{-4-7}{-1}, \frac{-2+1}{-2}) = \frac{1}{2} \Rightarrow \underline{\lambda}^T = \begin{bmatrix} -2.5 & -6 & 0 \end{bmatrix}$ and x_4 should come in.

Problem 1

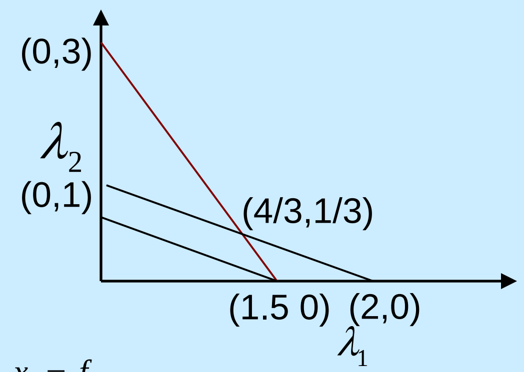
$$\underline{\lambda}^T = [-2.5 \quad -6.5 \quad -0.5]; B = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & 1 & 0 \\ -1.5 & 0 & 1 \end{bmatrix}$$

$$\underline{x}_B = B^{-1}\underline{b} = \begin{bmatrix} 1.5 \\ 2.5 \\ 16.5 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix};$$ optimal primal cost=-1.5-2.5*6=-16.5

optimal dual cost=-16.5 => optimal

$\underline{p}^T = \underline{c}^T - \underline{\lambda}^T A = [12.5\ 10.5\ 0.5\ 0\ 0\ 0] \Rightarrow$ satisfies CS conditions

Problem 2



Primal: $\max 2x_1 + 3x_2 + 3x_3$

$x_1 + 2x_2 + 2x_3 \le 12$

$2x_1 + 4x_2 + x_3 \le f$

$x_i \ge 0, i = 1,2,3$

Dual: $\min 12\lambda_1 + f\lambda_2$

$\lambda_1 + 2\lambda_2 \ge 2$

$2\lambda_1 + 4\lambda_2 \ge 3$

$2\lambda_1 + \lambda_2 \ge 3$

$\lambda_1 \ge 0; \lambda_2 \ge 0$

$0 \le f \le 6 \Rightarrow$ optimal dual solution: $[\lambda_1\ \lambda_2] = [0\ 3] \Rightarrow profit = 3f; x_1 = 0, x_2 = 0, x_3 = f$

$6 \le f \le 24 \Rightarrow$ optimal dual solution: $[\lambda_1\ \lambda_2] = [4/3\ 1/3] \Rightarrow profit = 16 + (1/3)f; x_1 = \frac{2(f-6)}{3}, x_2 = 0, x_3 = \frac{24-f}{3}$

$f \ge 24 \Rightarrow$ optimal dual solution: $[\lambda_1\ \lambda_2] = [2\ 0] \Rightarrow profit = 24; x_1 = 12, x_2 = 0, x_3 = 0$

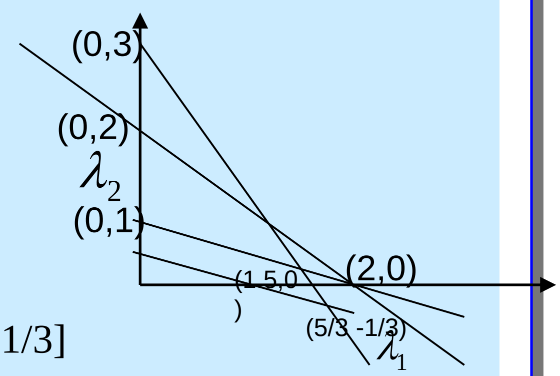
- Problem 3:

$$\min \quad 2x_1 + 3x_2 + 2x_3 + 2x_4$$
$$s.t. \quad x_1 + 2x_2 + x_3 + 2x_4 = 3$$
$$x_1 + x_2 + 2x_3 + 4x_4 = 5$$
$$x_i \geq 0,\ i = 1, 2, 3, 4$$

$$\max \quad 3\lambda_1 + 5\lambda_2$$
$$\lambda_1 + \lambda_2 \leq 2;\ 2\lambda_1 + \lambda_2 \leq 3$$
$$\lambda_1 + 2\lambda_2 \leq 2;\ 2\lambda_1 + 4\lambda_2 \leq 2$$

$$B = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} \Rightarrow \underline{x}_B = \begin{bmatrix} 1/3 \\ 7/6 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \Rightarrow \underline{\lambda}^T = \begin{bmatrix} 5/3 & -1/3 \end{bmatrix}$$

$$\underline{p}^T = \underline{c}^T - \underline{\lambda}^T A = [2/3\ 0\ 1\ 0] \Rightarrow \text{satisfies CS conditions}$$



For max 3 λ_1+5λ_2, optimal corner point in the dual space : [5/3 -1/3]

$\Rightarrow \underline{x}_B = [x_2\ \ x_4] = [1/3\ 7/6]$, cost=10/3

For max 8λ_1+7λ_2, optimal corner point does not change

$\Rightarrow \underline{x}_B = [x_2\ \ x_4] = [3\ \ 1]$; cost = 11

- Problem 4:

$$\min \quad 2x_1 + x_2 + 4x_3$$
$$s.t. \quad x_1 + x_2 + 2x_3 = 3$$
$$2x_1 + x_2 + 3x_3 = 5$$
$$x_i \geq 0, i = 1,2,3$$

Dual:

$$\max \; 3\lambda_1 + 5\lambda_2$$
$$\text{s.t.} \quad \lambda_1 + 2\lambda_2 \leq 2$$
$$\lambda_1 + \lambda_2 \leq 1$$
$$2\lambda_1 + 3\lambda_3 \leq 4$$

Iteration 0: $\underline{\lambda}^T = [1\,0]; P = \{2\}$

RP: $\min \underline{e}^T \underline{y}$

$$\text{s.t.} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 + \underline{y} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

DRP: $\max \; 3\mu_1 + 5\mu_2$

$$\text{s.t.} \quad \mu_1 + \mu_2 \leq 0; \mu_1 \leq 1; \mu_2 \leq 1$$

$$\Rightarrow \mu_1 = -1; \mu_2 = 1$$

$$\underline{p}^T = \underline{c}^T - \underline{\lambda}^T N = [1\,2]$$

$$\underline{\mu}^T N = [1 1] \Rightarrow \in = \min(1,2) = 1$$

$$\underline{\lambda}^T \leftarrow \underline{\lambda}^T + \in \underline{\mu}^T = [0 \;\; 1]$$

Iteration 1: $\underline{\lambda}^T = [0\,1]; P = \{1,2\}$

RP: $\min \underline{e}^T \underline{y}$

$$\text{s.t.} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 + \underline{y} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

DRP: $\max \; 3\mu_1 + 5\mu_2$

$$\text{s.t.} \quad \mu_1 + \mu_2 \leq 0; \mu_1 + 2\mu_2 \leq 0; \mu_1 \leq 1; \mu_2 \leq 1$$

$$\Rightarrow \mu_1 = 0; \mu_2 = 0$$

$$\underline{x}_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \text{ optimal cost=5}$$

- Problem 5: Text 9.1

$$\min\ f = x_1 + x_2 - \mu\sum_{i=1}^{3}\ln x_i$$

$$\text{s.t.}\quad x_1 + x_2 + x_3 = 1 \Rightarrow x_3 = 1 - x_1 - x_2$$

$$\Rightarrow \min\ f = x_1 + x_2 - \mu(\ln x_1 + \ln x_2 + \ln(1 - x_1 - x_2))$$

$$\nabla f = \begin{bmatrix} 1 - \dfrac{\mu}{x_1} + \dfrac{\mu}{1 - x_1 - x_2} \\ 1 - \dfrac{\mu}{x_2} + \dfrac{\mu}{1 - x_1 - x_2} \end{bmatrix} = \underline{0} \Rightarrow \text{ By symmetry } x_1 = x_2$$

$$\Rightarrow solution: x_1 = x_2 = \frac{1 + 3\mu - \sqrt{(1 + 9\mu^2 - 2\mu)}}{4}; x_3 = 1 - 2x_1$$