**Dept. of ECE** **Spring 2004**
**Univ. of Conn.** **KRP**

**<u>Homework Set # 2 (Due February 11, 2004)</u>**

1. Suppose that whether or not it snows this week depends on weather conditions through the last two weeks. Specifically, suppose that if it has snowed for the past two weeks, then it will snow next week with probability 0.7; if it snowed this week but not last week, then it will snow next week with probability 0.5; if it snowed last week but not this week, then it will snow next week with probability 0.4; it if has not snowed in the past two weeks, then it will snow next week with probability 0.2. Draw the state transition diagram and determine the transition probability matrix. Obtain the steady-state probability vector, if it exists (Hint: Need four states).
2. A transition probability matrix $P$ is termed *doubly stochastic* if both its row and column sums are unity (i.e., $\|P\|_\infty = \|P\|_1=1$). If a Markov chain with doubly stochastic $P$ is irreducible, aperiodic, and finite with n states (i.e., its steady state probability distribution exists), find its steady state probability vector.
3. Consider the multi-processor memory interference problem with two processors and two memory modules considered in the class. Suppose that processor 1 requests access to memory module 1 with probability $r_1$ and memory module 2 with probability $r_2$ (1- $r_1$), while processor 2 requests access to memory module 1 with probability $q_1$ and memory module 2 with probability $q_2$ =(1- $q_1$). Assume that the memory access times are constant, and each processor generates memory access requests immediately after the previous request has been satisfied.
   a. Construct a Markov chain state diagram (Hint: Each state is a 4-tuple denoting the number of each processor requests at each memory module).
   b. Solve for the steady-state probabilities of the Markov chain.
   c. Compute the average number of memory requests completed per memory cycle, *E(B)*.
   d. Solve the following optimization problem: $\underset{r_1, q_1}{Max}\, E(B)$
4. Consider a data structure (e.g., a linear array) being manipulated within a program. Suppose we are interested in the amount of memory used by the data structure. If the current memory in use is *i* units, then we say that the state of the structure is $s_i$. Let the probabilities associated with the next operation on the data structure be given by:

   $b = P\{\text{next operation is an insert (i.e., state } s_{i+1} \text{ with } (i+1) \text{ units of memory)} \mid \text{current state } s_i)$

   $d = P\{\text{next operation is a delete (i.e., state } s_{i-1} \text{ with } (i-1) \text{ units of memory)} \mid \text{current state } s_i)$

   $a = P\{\text{next operation is an access (i.e., state } s_i \text{ with i units of memory)} \mid \text{current state } s_i)$

   $= (1 - b - d)$

   (i) Assuming infinite memory, and $d > b$, draw the Markov-chain state diagram and show that the steady-state probabilities $\{p_i : i = 0,1,2,..\}$, where $p_i$ is the probability that the memory in use is $i$ units (i.e., state $s_i$) is $(1-\frac{b}{d})(\frac{b}{d})^i$, and the average number of memory units in use, $Q$ is $\frac{b}{(b-d)}$. (Hint: This is a discrete-time birth-death process with a tri-diagonal transition probability matrix).

(ii) Suppose we have only a finite amount of memory, say $M$ units. Assume that an insertion operation is ignored when all $M$ units are in use, leaving the memory state in $s_M$. Find the steady-state probabilities in this case.

5. Let $\tau$ be a nonnegative random variable with distribution function $H_\tau(t)$ and density $h_t(t)$ . Let $E(\tau) = \dfrac{1}{\lambda}$, $0 < \lambda < \infty$. Prove that the following three statements are equivalent:
   a. τ has exponential distribution: $H_\tau(t) = 1 - e^{-\lambda t}$.
   b. τ has memory-less property: $P(\tau > t + r \mid \tau > r) = P(\tau > t)$ for all $t$ and $r$ >0.
   c. τ has the constant hazard rate property: $\dfrac{h_\tau(t)}{1 - H_\tau(t)} = \lambda$, a constant for all $t$>0.
6. Consider a packet stream whereby packets arrive according to a Poisson process with rate 10 packets per second. If the inter-arrival time between any two successive packets is less than the transmission time of the first, the two packets are said to "collide". Find the probability that a packet collides with either its predecessor or its successor assuming:
   a. All packets have a constant transmission time of 20 milliseconds.
   b. Packets have independent, exponentially distributed transmission time with mean 20 milliseconds.
7. Variations on Poisson processes
   a. Suppose that in a pure birth process, the birth rate $\lambda(t)$ is time-varying. Such a process is termed a non-homogeneous Poisson process. Show that $p_n(t)$ is given by:
   $$p_n(t) = \frac{[\int_0^t \lambda(\tau)d\tau]^n}{n!} e^{-\int_0^t \lambda(\tau)d\tau} \ ; n = 0,1,2,....$$
   b. Suppose that the birth rate is a function of $n$ only, i.e., $\lambda(n)$. Show that $p_n(t)$ is given by:
   $$p_n(t) = e^{-\lambda(n)t} p_n(0) + \lambda(n-1) \int_0^t e^{-\lambda(n)(t-\tau)} p_{n-1}(\tau)d\tau$$
   c. Suppose that the birth-rate in a pure birth process $\lambda(n) = n\,\lambda$ and the initial population $X(0)$ =1, i.e., $p_1(0) = 1$ and $p_n(0) = 0$ for all $n \neq 1$. This is a special case of *Yule-Furry* process. Show that the differential equation for $p_n(t)$ is given by:
   $$\frac{dp_n(t)}{dt} = (n-1)\lambda\ p_{n-1}(t) - n\lambda\ p_n(t),\ n \geq 1$$
   Solve for $p_n(t)$. (Hint: use induction.)
8. The nonnegative random variables $\{\tau_i : i = 1,2,..\}$ are i.i.d. random variables with the density function $h_\tau(t)$ and Laplace transform $L_\tau(s)$. Consider a Poisson process $N(t)$ with parameter $\lambda$ which is independent of the random variables $\{\tau_i : i = 1,2,..\}$. Consider now a second random process of the form: $Y(t) = \sum_{i=1}^{N(t)} \tau_i$. Show that $L_Y(s,t) = \exp(\lambda\ t\,[L_\tau(s) - 1])$. (Hint: Use the total probability theorem).