Dept. of ECE **Spring 2004**
Univ. of Conn. **KRP**

Homework Set # 3 (Due February 18, 2004)

1. Properties of Poisson Distribution
   a. Show that the superposition of *m* independent Poisson streams with rates $\{\lambda_i\}_{i=1}^m$ results in a Poisson stream with rate $\lambda = \sum_{i=1}^{m} \lambda_i$.
   b. Show that the decomposition of a Poisson stream with rate λ into *m* independent streams, where each stream is chosen independently with probability $\{p_i\}_{i=1}^m$ results in *m* Poisson streams with rates $\{p_i\lambda\}_{i=1}^m$.
   c. Suppose we know that in an interval $[t_1, t_2]$ only one arrival of a Poisson process has occurred. Show that, conditional on this knowledge, the time of this arrival is uniformly distributed in $[t_1, t_2]$.
   d. Given that the number of arrivals *X(t)* = *n*, show that the *n* arrival times $\{S_i\}_{i=1}^n$ have the same distribution as *order statistics* corresponding to *n* independent random variables uniformly distributed on the interval $[0, t]$.
   e. Consider a system in which customers at any time are classified as being in one of *r* possible classes, and assume that a customer changes class in accordance with a Markov chain having transition probabilities $P = [P_{ij}]_{i,j=1}^r$. That is, if a customer is in class *i* during a time period then, independently of his previous classes, he will be in class *j* during the next time period with probability P_{ij}. The customers are assumed to move through the system independently of each other. Suppose that the number of customers initially in states *1,2,..,r* are independent Poisson random variables with respective means $\{\lambda_i\}_{i=1}^r$. Determine the joint distribution of the numbers of customers in states *1,2,…,r* at some time *n*.
   f. A stochastic process $\{Z(t), t \geq 0\}$ is said to be a compound Poisson process if it can be represented as $Z(t) = \sum_{i=1}^{X(t)} \tau_i$ where $\{X(t), t \geq 0\}$ is a Poisson process and $\{\tau_i, i \geq 1\}$ are i.i.d. random variables with mean $\bar{\tau}$ and variance σ_τ^2. Find the mean and variance of *Z(t).*
2. Consider a birth-death queuing system in which $\lambda(n) = \lambda$ and $\mu(n) = n\mu$. For all n, find the differential-difference equations for $p_n(t)$, the probability that there are n customers in the system. Find the partial differential equation for the evolution of *G(z,t)*, the z-transform of $p_n(t)$. Show that: $G(z,t) = \exp(\frac{\lambda}{\mu}(1 - e^{-\mu t})(z-1))$. Recalling its similarity to the z-transform of the Poisson distribution, find $p_n(t)$. What happens to $p_n(t)$ as $t \to \infty$.

3. Let X *and* Y be independent exponential random variables with respective means $1/\lambda_1$ *and* $1/\lambda_2$. Further, let $Z = \min(X, Y)$. What is the conditional distribution of *Z* given that *Z=X*. Also, argue why the conditional distribution of *Y-Z*, given that *Z=X*, is exponential with mean $1/\lambda_2$.
4. Let X *and* Y be independent exponential random variables, each with mean $1/\mu$.. Further, let $Z = \min(X, Y)$, *and* $W = \max(X, Y)$. Find *E(Z), E(W), Var(Z) and Var(W).*