The University of Connecticut **Fall 2008**
Dept. of ECE **KRP**

Problem set # 1

REVIEW of LINEAR ALGEBRA

(Due September 3, 2008)

1. For a symmetric matrix *A* with distinct eigen values, show that:
   (a) All eigen values must be real
   (b) Eigen vectors associated with distinct eigen values are orthogonal
   (c) The matrix can be diagonalized using an orthogonal transform

2. The Rayleigh quotient for a symmetric matrix is defined as:
   $$r = \frac{x^T A x}{\underline{x}^T \underline{x}}$$
   Show that
   $$\lambda_{\min}(A) \leq r \leq \lambda_{\max}(A)$$

3. What is the condition number (using the *p*=2 norm) of a n orthogonal matrix *Q*? Show that the eigen values of an orthogonal matrix satisfy $|\lambda_i|=1$.

4. Show that if *S* is skew symmetric (i.e., $S^T = -S$), then $(I–S)$ is nonsingular and $(I–S)^{-1}(I+S)$ is orthogonal. (This is known as the Caley transformation of *S*.)

5. If *A* and *B* are $n \times r$ and $r \times m$ matrices, respectively, show that the product $C = AB$ can be written as a dyadic summation of the form
   $$C = \sum_{i=1}^{r} \underline{a}_i \underline{b}_i^T \text{ AND}$$
   where $\underline{a}_i$ is the *i*th column of A and $\underline{b}_i^T$ is the *i*th row of *B*. Thus, show that *C* may be recursively computed via
   $$C_i = C_{i-1} + \underline{a}_i \underline{b}_i^T; \qquad i = 1, 2, \ldots, r$$
   with the initialization $C_0 = [0]$. Comment on the efficient use of this scheme when *A* is “sparse”.

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9. Approximately how many multiplies and adds (MADDS) are needed to multiply two $n \times n$ upper triangular matrices (a lower triangular matrix has $a_{ij} = 0$ for $j>1$)? How many MADDS are needed to form $C = AB$ where *A* is an upper triangular matrix and *B* is a lower triangular matrix.

6. If $\underline{a}$ and $\underline{b}$ are *n* vectors, what are the eigen values of the matrix $C = \underline{ab}'$?

7. An *n* x *n* matrix *A* is said to be elementary if it differs from the identity matrix by a matrix of rank one, i.e., $A = I - \underline{x}\,\underline{y}^T$ for some vectors $\underline{x}$ and $\underline{y}$. Under what conditions on $\underline{x}$ and $\underline{y}$, does *A* have an inverse. Show that $A^{-1} = I - \sigma \underline{x}\,\underline{y}^T$ for some scalar σ.

8. For an $m \times n$ matrix A, prove the following relations:

(a) $\rho(A) \le \|A\|_p$ for any p-norm; $\rho(A) =$ spectral radius of A

(b) $\max_{i,j} a_{ij} \le \|A\|_2 \le (mn)^{1/2} \max_{i,j} a_{ij}$

(c) $\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}| =$ maximum column sum

(d) $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| =$ maximum row sum

(e) $\|A\|_2 \le \left[\|A\|_1 \|A\|_\infty\right]^{1/2}$

(f) $(n)^{-1/2} \|A\|_\infty \le \|A\|_2 \le (m)^{1/2} \|A\|_\infty$

(g) $(m)^{-1/2} \|A\|_1 \le \|A\|_2 \le (n)^{1/2} \|A\|_1$

(h) For all orthogonal matrices Q and Z of appropriate dimension, the Frobenius and 2-norms satisify:

$$\|QAZ\|_F = \|A\|_F \text{ and } \|QAZ\|_2 = \|A\|_2$$

9. True/False Questions

a. For a symmetric matrix A, it is always the case that $\|A\|_1 = \|A\|_\infty$.
b. If a triangular matrix has a zero entry on its main diagonal, then the matrix is necessarily singular.
c. The product of two symmetric matrices is symmetric.
d. A symmetric positive definite matrix is always well-conditioned.
e. $\|A\|_1 = \|A^T\|_\infty$.
f. Can a system of linear equations $A\underline{x} = \underline{b}$ have exactly two distinct solutions?
g. If $\underline{x}$ is any vector, then $\|\underline{x}\|_1 \ge \|\underline{x}\|_\infty$
h. If A is any n x n nonsingular matrix, then $\kappa(A) = \kappa(A^{-1})$.
i. A system of linear equations $A\underline{x} = \underline{b}$ has a solution if and only if the m x n matrix A and the augmented m x ($n+1$) matrix [A $\underline{b}$] have the same rank.
j. The product or inverse of an upper triangular matrix is upper triangular.

10. Classify each of the following matrices as well-conditioned or ill-conditioned:

$$(a)\ A = \begin{bmatrix} 10^{10} & 0 \\ 0 & 10^{-10} \end{bmatrix}; \quad (b)\ B = \begin{bmatrix} 10^{10} & 0 \\ 0 & 10^{10} \end{bmatrix}$$

$$(c)\ C = \begin{bmatrix} 10^{-10} & 0 \\ 0 & 10^{-10} \end{bmatrix}; \quad (d) D = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$