Problem Set # 11

SYMMETRIC EIGEN VALUE PROBLEM

(Due Nov 19 2008)

(Do only problems 1 and 2)

1. Suppose A and $A+E$ are n x n symmetric matrices. Then, for $k = 1, 2,\ldots,n$, show that:

   $\lambda_k(A)+\lambda_n(E) \leq \lambda_k(A+E) \leq \lambda_k(A)+\lambda_1(E)$

   Assume $\lambda_n \leq \lambda_{n-1} \leq \ldots \leq \lambda_1$. Use this to show that $\lambda_i(A+\tau \underline{c}\, \underline{c}^T) = \lambda_I(A)+m_i\tau$ where $m_1+m_2+\ldots m_n=1$, $m_i \geq 0$, $\underline{c}$ has unit 2-norm.

2. Suppose A is symmetric and positive definite. Consider the following iteration:

   $A_0=A$

   $For\ k=1,2,\ldots$

   $A_{k-1} = L_k L_k^T$ *(Cholesky)*

   $A_k = L_k^T L_k$

   Show that this iteration provides similarity transformations and work out a 2 x 2 example to show that A_k converges to Diag (λ_i).

3. Suppose A has the special form $(I+\underline{v}\,\underline{v}^T)$ where $\underline{v}$ is a combination $c\underline{q}^1 + d\underline{w}$ of unit vectors.
   (a) If the Lanczos algorithm starts with $\underline{q}_1$, show that its next vector is $\underline{q}_2$ is $\underline{w}$.
   (b) Compute $a_1 = \underline{q}_1^T A\, \underline{q}_1$, $b_1=\underline{q}_2^T A\underline{q}_1$, $a_2= \underline{q}_2^T A\, \underline{q}_2$ in terms of c and d.
   (c) Show that the 2 by 2 matrix T_2 with these entries has the same Eigen values 1 and $(1 + c^2 + d^2)$ as A itself.

4. Show that $A\underline{q}_i$ is orthogonal to $\underline{q}_j$ when $i \leq (j-2)$, by using the two Lanczos properties: $\underline{q}_i$ is a linear combination of $\underline{q}_1, \ldots, A^{i-1}\underline{q}_1$, and $\underline{q}_j$ is orthogonal to all combinations of $\underline{q}_1, \ldots, A^{j-1}\underline{q}_1$.

5. Show from the Lanczos recursion that $\underline{q}_{j+1}$ is orthogonal to $\underline{q}_j$ if $a_j = \underline{q}_j^T A\, \underline{q}_j$. Further, show that $\underline{q}_{j+1}$ is also orthogonal to $\underline{q}_{j-1}$ for $i = j-1$.