The University of Connecticut Fall 2008
Dept. of ECE KRP

Problem Set # 13

(Due Nov 19, 2008)

(Do problems 1, 2, 3 and 5 only)

1. Solve $A^T X + XA = I$, when

$$A = \begin{bmatrix} -0.15365 & 0.0040173 & 0.17786 & -0.99009 & 0.075158 & 0.0 & 0.0 & 0.0 \\ -1.2482 & 2.8543 & 0.0 & 1.4324 & -0.72689 & 4.0383 & 0.0 & 0.0 \\ 0.0000 & 1.0000000 & 0.00000 & 0.00000 & 0.000000 & 0.0 & 0.0 & 0.0 \\ 0.56788 & -0.27685 & 0.0 & -0.28366 & -2.0496 & -0.13886 & 0.0 & 0.0 \\ 0.00000 & 0.000000 & 0.00000 & 0.0000 & -10.00000 & 0.0 & 0.00000 & 0.0 \\ 0.00000 & 0.000000 & 0.0 & 0.0 & 0.0 & -20.000 & 0.0 & 0.0000 \\ 0.00000 & 0.000000 & 0.0000 & 0.00000 & 0.000000 & 0.0 & -3.00000 & -2.0 \\ 0.00000 & 0.000000 & 0.0 & 0.0 & 0.0 & 0.0000 & 1.0 & 0.0000 \end{bmatrix}$$

2. Consider a second order system

$$\dot{X}_1 = X_2$$
$$\dot{x}_2 = -x_1 - 2\zeta x_2$$

with the initial conditions $x_1(0)$ and $x_2(0)$. Find the optimal value of the damping ration ζ to minimize the quadratic cost function:

$$J = \int_0^\infty (\underline{x}^T Q \underline{x}) dt$$

where $Q = Diag(1, \mu)$. What is the optimal ζ when $x_2(0) = 0$. What is the optimal ζ when $x_2(0) = 0$ and when (i) $\mu = 0$, and (ii) $\mu = 1$.

3. One of the problems with the optimization in problem 2 is that the optimal ζ depends on the initial conditions $x_1(0)$ and $x_2(0)$. In order to remove this dependence, it is suggested that we optimize:

$$\tau = \max_{\underline{x}(0)} \left\{ \min_{\zeta} \frac{\underline{x}^T(0) P \underline{x}(0)}{\underline{x}^T(0) \underline{x}(0)} \right\}$$

where *P* is the solution of the Lyapunov equation. You can think of this as a game between nature and human designer: nature tries to maximize the initial state, while the human minimize the time constant. Show (after lengthy calculations) that when $\mu = 1$, the optimal parameters are:

$$x_1^*(0) = 0.899 \| \underline{x}(0) \|_2$$
$$x_2^*(0) = 0.437 \| \underline{x}(0) \|_2$$
$$\zeta^* = 0.786$$
$$\tau^* = 1/2(\zeta^* + 1/\zeta^* + [\zeta^{*2} + 1]^{1/2}) = 1.665$$

This agrees with the rule of thumb that the optimal damping ratio should be approximately 0.7.

4. Consider a linear dynamic system:

$$\dot{\underline{x}} = A\underline{x}(t) + B\underline{u}(t)$$

where the control variables are subject to the constraints:

$$| u_i(t) | < a_i < \infty, \; i = 1, 2, ..., m$$

It is well known from the so-called second theorem of Lyapunov that if $V(\underline{x}(t), t) = \underline{x}^T(t)P, \; \underline{x}(t)$ is a Lyapunov function and $dV(\underline{x}(t), t)/dt < 0$, the system is asymptotically stable. So, a natural criterion for optimization is to make $dV(\underline{x}(t), t)/dt$ as negative as possible. Show that the optimal control is the bang-bang control given by:

$$u_i(t) = -a_i \operatorname{sgn}(B^T P \underline{x}(t))_i$$

where *sgn x* is given by:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

One problem with the bang-bang control is this: When the optimal control is substituted into the state equation, there is no guarantee that the resulting differential equation has a solution, since *sgn(x)* is discontinuous. This difficulty may be avoided by rewriting the control equation as:

$$u_i(t) = -a_i \, \mathrm{s}at[k(B^T P \underline{x}(t))_i]$$

where

$$sat(x) = \begin{cases} 1 & \text{if } x > 1 \\ x & \text{for } |x| \le 1 \\ -1 & \text{if } x < -1 \end{cases}$$

By selecting *k* as large as desired, we can approximate the *sgn(x)* function arbitrarily closely. In order to test the above theory, consider the plant specified by:

$$A = \begin{bmatrix} -0.01 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2.26 & -0.2 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 0 \\ 2.5 \end{bmatrix}$$

Using $Q = I$, solve the Lyapunov equation $A^T P + PA = -Q$, and use $V(\underline{x}(t), t) = \underline{x}^T(t) P \underline{x}(t)$ as a Lyapunov function. Assume that $a_i = 1$.

(a) Compute the responses of the system with no control (i.e., $u(t) = 0$ for all t) and two initial conditions: $(i)\ \underline{x}(0) = (0,0,10)^T$, (ii) $\underline{x}(0) = (0,0,3)^T$
(b) Implement the optimal bang-bang control for the two initial conditions
(c) Implement the approximation to bang-bang control using *k* = 1, 10, 100, 1000, 10000.
(d) Discuss results of (a-c).

5. Consider a linear time-variant system

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$
$$\underline{y}(t) = C\underline{x}(t)$$

The objective is to find an output feedback control law of the form:

$$\underline{u}(t) = -L\underline{y}(t)$$

such that the following quadratic cost function is minimized:

$$J = \frac{1}{2}\int_0^\infty [\underline{x}^T(t) Q \underline{x}(t) + \underline{u}^T(t) R \underline{u}(t)] dt$$

Show that the optimal feedback gains *L** satisfy the following necessary conditions of optimality:

$$0 = (A - BL^*C)^T P + P(A - BL^*C) + Q + C^T L^T RLC$$
$$0 = (A - BL^*C)\Lambda + \Lambda(A - BL^*C)^T + \underline{x}(0)\underline{x}(0)^T$$
$$L^* = R^{-1} B^T P \Lambda C^T [C^T AC]^{-1}$$

Note: in order to remove the dependence of the optimal control on the initial state $\underline{x}(0)$, it is often assumed that $\underline{x}(0)$ is random with mean $\underline{m}$ (typically 0) and covariance $\sum$ (typically, identity matrix). In this case, $\underline{x}(0)\underline{x}(0)^T$ in the Λ equation is replaced by $(\underline{m}\underline{m}^T + \sum\ )$.