The University of Connecticut **Fall 2008**
Dept. of ECE **KRP**

Problem Set # 2
(Due September 10, 2008)
Computation of e^{AT}

Part I: Computational

Extensively test the accuracy of the subroutine **expm** of MATLAB, which uses Pade approximation for computing e^{AT}. You are to devise the test cases and discuss the results. Specifically:

1. Consider matrices of order up to n=6.
2. Consider positive definite, positive-semi-definite, negative definite, negative-semi-definite, and indefinite matrices with clustered and widely separated eigen values.
3. Devise a way to assess the accuracy of the results (e.g., by (1) knowing the correct answer by hand calculation or special matrices for which you know the answer (e.g., idempotent and skew-symmetric matrices), (2) computing e^{-AT} and testing $e^{-AT} * e^{AT} = I_n$, (3) $\det(e^{AT}) = e^{\mathrm{trace}(AT)}$. Read the references for additional ideas.

Part II: Analytical

1. Compute the coefficients of a fourth order Chebyshev approximation to e^a, Substitute for $T_i(a)$, i = 0, 1, 2, 3, 4 their polynomial representation to obtain power series:
$$e^a \approx \sum_{i=0}^{4} b_i a^i = f(a)$$
How do the coefficients b_i compare to those of a Taylor series. Plot the error curve $[e^a - f(a)]$ over the range $-1 \le a \le 1$.
2. Compute the 2nd order Pade approximation to e^a:
$$f(a) = \frac{\left(1 + \frac{a}{2} + \frac{a^2}{12}\right)}{\left(1 - \frac{a}{2} + \frac{a^2}{12}\right)}$$
with the Taylor series by seeing how many terms in a^i the two equivalent series agree (note that $f(a)$ can be obtained by dividing the numerator by the denominator.) What is the coefficient of the first error term? How does it compare with that of a fourth order Chebyshev approximation?
   Derive Clenshaw recursion for Chebyshev function approximation.
   See Notes
3. (a) Show that $e^{(A+B)T} = e^{AT} \cdot e^{BT}$ if and only if $AB = BA$.

   (b) Show that $\lim_{t\to\infty} e^{AT} \to 0$ if and only if $\mathrm{Re}\{\lambda_i(A)\} < 0$.

(c) Show that if A is skew-symmetric, then e^A is orthogonal.

(d) Show that for any n by n matrices A and E,

$$e^{(A+E)\tau} - e^{A\tau} = \int_0^\tau e^{A(\tau-s)} E e^{(A+E)s} ds$$

(Hint: $Let\, \Phi = e^{(A+E)t} \Rightarrow \dot{\Phi} = (A+E)\Phi = A\Phi + E\Phi; \Phi(0) = I$)

4. Show that for n x n matrices A and Q, if

$$\exp\left(\begin{bmatrix} -A^T & Q \\ 0 & A \end{bmatrix} T\right) = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

then

$$S_{22}^T S_{12} = \int_0^T e^{A^T t} Q e^{At} dt$$

This will be useful in Lecture 3.