The University of Connecticut **Spring 2016**
Dept. of ESE **KRP**

Homework Set # 3 (Due March 3, 2016)

1. Using the revised simplex procedure, solve:

$$\begin{aligned} &\textit{Maximize } 2x_1 + 4x_2 + x_3 + x_4 \\ &\textit{subject to } x_1 + 3x_2 + x_4 \le 4 \\ &2x_1 + x_3 \le 3 \\ &x_2 + 4x_3 + x_4 \le 3 \\ &x_i \ge 0,\ i = 1,2,3,4 \end{aligned}$$

   a) How much can the elements of $\underline{b}^T = (4\ 3\ 3)$ be changed without changing the optimal basis?

   b) How much can the elements $\underline{c}^T = (2\ 4\ 1\ 1\ )$ be changed without changing the optimal basis?

   c) What happens to the optimal cost for small changes in $\underline{b}$?

   d) What happens to the optimal cost for small changes in $\underline{c}$?

2. Exercise 3.21 from Text, Page 134

3. Consider an assignment problem of allocating certain activities i ($1 \le i \le n$) to individuals j ($1 \le j \le n$). If activity i is allocated to individual j, a value of s_{ij} units (dollars) is generated. The objective is to maximize the total value generated, subject to the constraint that at most one activity can be allocated to an individual, and that all activities must be allocated. The problem, also known as the weighted bipartite matching problem, can be formulated as:

$$\max \quad \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}x_{ij}$$

$$s.t.$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \textit{for } i = 1,2,..,n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \textit{for } j = 1,2,..,n$$

$$x_{ij} = 0, 1$$

   a. Write the dual problem formulation.
   b. Show that there exist dual prices λ_i ($1 \le i \le n$) and μ_j ($1 \le j \le n$) such that $\lambda_i + \mu_j \ge s_{ij}$ for all i and j with equality holding if in an optimal assignment an activity i is assigned to individual j.

c. Show that part (b) implies that if activity i is optimally assigned to individual j and if k is any other individual $s_{ij} - \mu_j \geq s_{ik} - \mu_k$. Give an economic interpretation of this result.

4. (a) Assume A is a square matrix. Obtain sufficient conditions on **c**, A and **b** under which the LP problem

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ s.t. \quad & A\mathbf{x} \leq \mathbf{b}; \mathbf{x} \geq \mathbf{0} \end{aligned}$$

and its dual are identical.

(b) Given the following LP problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} i x_i \\ s.t. \quad & \sum_{i=k}^{n} x_i \leq 1 \;\; for k = 1,2,..,n \\ & x_i \geq 0 \quad for\ all\ i \end{aligned}$$

Consider its dual problem and determine the optimal solution without any numerical computation.

5. Find the dual of

$$\begin{aligned} \min \quad & \underline{c}^T \underline{x} \\ s.t. \quad & A\underline{x} \leq \underline{b}; \underline{x} \geq \underline{a} \ where\ \underline{a} \geq 0 \end{aligned}$$

6. Let A be a m x n matrix and $\underline{c}$ be an n-vector. Prove that $A\underline{x} \leq \underline{0}$ implies $\underline{c}^T \underline{x} \leq \underline{0}$ if and only if $\underline{c}^T = \underline{\lambda}^T A$ for some $\underline{\lambda} \geq \underline{0}$. Give a geometric interpretation of the result.

7. Consider the primal LP: $\min \quad \underline{c}^T \underline{x}\ s.t.\ A\underline{x} = \underline{b}; \underline{x} \geq 0$. Suppose that this problem is dual and primal feasible. Let $\underline{\lambda}$ be a known optimal solution to the dual.

   a. If the k^{th} constraint equation of the primal is multiplied by $\mu \neq 0$, determine an optimal solution $\underline{w}$ to the dual of this new problem.
   b. Suppose that, in the original primal, we add μ times the k^{th} constraint equation to the r^{th} constraint equation. What is an optimal solution $\underline{w}$ to the corresponding dual problem?
   c. Suppose, in the original primal, we add μ times the k^{th} row of A to $\underline{c}$. What is an optimal solution $\underline{w}$ to the corresponding dual problem?

8. Exercise 4.7 of Text, pp. 189.
9. Exercise 5.2 of Text, pp.222.
10. Exercise 5.4 of Text, pp. 223.

Computational (Due March 10,2016). This can be done with a team of at most 3.

Implement Revised Simplex and primal-dual path following interior point algorithm with Mehrotra's predictor-corrector method in MATLAB. Textbook is an excellent source for interior point algorithms. Extensively test the algorithms with a variety of test cases gleaned from the literature. The Web has a number of sites where you can get test cases. One example: http://neos-guide.org/content/linear-programming