University of Connecticut **Fall 2008**
ECE Dept. **KRP**

Problem Set # 6

(Due October 8, 2008)

ORTHOGONALIZATION METHODS

Part A: Analytical (Do only 1, 2, 4, 5, 9, 10)

1. Compute the QR decomposition of:

$$A = \begin{bmatrix} 5 & 9 \\ 12 & 7 \end{bmatrix}$$

   using (a) Householder and (b) Gram-Schmidt orthogonalization procedures.

2. Factor the following matrix into $A=QR$:

$$A = \begin{bmatrix} \frac{2}{3} & 0 \\ \frac{2}{3} & 2 \\ \frac{1}{3} & -1 \end{bmatrix}$$

3. If A contains the first n columns of the m by m identity matrix, what is the best least-squares solution to $A\underline{x}=\underline{b}$ where $\underline{x} \in R^n$ and $\underline{b} \in R^n$.

4. The squared Euclidean distance from the point $\underline{b}$ to the line through $\underline{a}$ is

$$\| \underline{b} - \underline{p} \|^2 = \| \underline{b} - \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \underline{a} \|^2 = \frac{(\underline{b}^T \underline{b})(\underline{a}^T \underline{a}) - (\underline{a}^T \underline{b})^2}{\underline{a}^T \underline{a}}$$

   Since this cannot be negative, the numerator gives the Schwartz inequality:

$$(\underline{a}^T \underline{b})^2 \leq (\underline{b}^T \underline{b})(\underline{a}^T \underline{a})$$

   (a) How does this inequality come from:

$$\underline{a}^T \underline{b} = ||a||_2 ||b||_2 \cos\theta$$

   (b) By making the right choice of $\underline{b}$, show that:

$$\left(\sum_{i=1}^{n} a_i \right)^2 \leq n \sum_{i=1}^{n} a_i^2$$

5. Find the best least squares solution to:

$$\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   What is the projection of $\underline{b}$ onto the space spanned by these two columns? What is its projection orthogonal to this space ?

6. When can the points x_1 and x_2 on the x-axis be the projections of two orthonormal vectors (x_1, y_1) and (x_2, y_2)? Draw a figure and prove that $(x_1^2+x_2^2) = 1$.
7. The projection matrix onto the column space of A is $P=A(A^TA)^{-1}A^T$. If $A=QR$, find a simpler formula for P.
8. Find the Householder matrix H_1 that produces zeros below the first entry in

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}$$

   Compute H_1A.
9. If $H=I-2\underline{u}\underline{u}^T$, show that the unit vector $\underline{u}$ is an eigen vector and find the eigen value. Show also that vector $\underline{x}$ perpendicular to $\underline{u}$ is an eigen vector with eigen value $\lambda=1$; in other words, $H\underline{x}=\underline{x}$. What is the determinant and trace of H ?
10. Consider an n by n matrix A such that $A=QR$. Show that $A_1=RQ$ is similar to A. This will be the basis of eigen value-eigen vector computation.

Part B: Computational (October 22, 2008)

Compare the accuracy of LU decomposition vs Householder orthogonalization for obtaining the solution of $A\underline{x}=\underline{b}$. Examine several test cases of your choice and present your results. Assume that A is $n\times n$.