University of Connecticut **Fall 2008**
ECE Dept. **KRP**

Problem Set #7
(Due October 22, 2008)

ORTHOGONALIZATION METHODS AND LEAST SQUARES

1. Most orthogonalization methods in numerical analysis form the orthogonal matrix Q by accumulating elementary transformations (e.g., Householder, Givens, Gram-Schmidt, etc.). As a result, final Q is not quite orthogonal due to round-off accumulation. Develop an improvement scheme that will such a given, not quite orthogonal, Q and make it more orthogonal. The orthogonality error should decrease to zero with quadratic convergence.
(Hint:Use $Q_{n+1} = Q_n + \alpha\left[I - Q_n Q_n^T\right]Q_n;\;\; Q_0 = Q, \alpha = 1/2$)

2. Suppose you know $C_1 = \left[S^T S\right]^{-1}$. Then, find $C = \left[R^T R\right]^{-1}$, where
$$R = \begin{bmatrix} \alpha & \underline{v}^T \\ 0 & S \end{bmatrix}$$

3. Generate 11 data points by taking $t_i = (i-1)/10$ and $b_i = erf\,(t_i), i = 1,2,....,11$. where:
$$erf\,(x) = \frac{2}{\sqrt{\pi}}\int_0^{\pi} \exp\left(-t^2\right).dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$
   (a) Fit the data in a least-squares sense with polynomials of degree from 1 to 10. Compare the fitted polynomial with $erf\,(t)$ for a number of values of t between the data points, and see how the maximum error depends on n, the number of coefficients in the polynomial.
   (b) Since $erf\,(t)$ is an odd function of t, that is, $erf\,(t) = -erf\,(-t)$, it is reasonable to fit the same data by a linear combination of odd powers of t,
   $$erf\,(t) \approx c_1 t + c_2 t^3 + ... + c_n t^{2n-1}$$
   Again, see how the error between data points depends on n. Since t varies over [0,1] in this problem, it is not necessary to consider using basis polynomials.
   (c) Polynomials are not particularly good approximations for $erf\,(t)$ because they are unbounded for large t. So, using the same data points, fit a model of the form:
   $$erf\,(t) \approx c_1 + e^{-t^2}\left(c_2 + c_3 z + c_4 z^2 + c_5 z^3\right)$$
   where $z = 1/(1+t)$. How does the error between the data points compare with the polynomial models ?

4. The following figures from the U.S. census Bureau give the population of the United states.

Year	Population
1900	75,994,575
1910	91,972,266
1920	105,710,620
1930	122,775,046
1940	131,669,275
1950	150,697,361
1960	179,323,175
1970	203,235,298
1980	226,547,082

5. An outlier can sometimes have a dramatic effect on the parameters in a model, and this is why it is important to check for such points, and determine if they are correct. To illustrate this point, we will consider the following artificial data set:

$$t_i = \begin{cases} i, & i = 1, 2, ..., 10 \\ 20, & i = 11 \end{cases} \quad ; \quad y_i = \begin{cases} 0, & i = 1, 2, ..., 10 \\ M, & i = 11 \end{cases}$$

We will experiment with various values of M. Fit the data to a straight line using least-squares approach for M = 0, 5, 10, 15, 20. How rapidly do the parameters change as a function of M ? Using a scaled residual plot (i.e., residual/its standard deviation), is it possible to identify point 11 as a potential outlier ?

6. Using weighted least squares, fit the following data by a quadratic polynomial:

t	y	σ_y
0.00	20.00	20.00
0.25	51.58	24.13
0.50	68.73	26.50
0.75	75.46	27.13
1.00	74.36	26.00
1.25	67.09	23.13
1.50	54.73	18.50
1.75	37.98	12.13
2.00	17.28	4.00

7. Many cyclic phenomena in nature can be modeled by a sinusoid and its higher harmonics:

$$y(t) = \sum A_i \sin\left[\frac{2\pi}{(T/i)} (t + \phi_i) \right]$$

The number A_i, T and ϕ_i are called the amplitude, period, and phase, respectively, of the i-th harmonic. Given data, fitting to this model in the least squares sense is a nonlinear least-squares problem. However, in many situations, the period T is known or can be guessed, leaving only the amplitude and phase as unknowns. However, this still appears to be nonlinear because ϕ_i is inside the sine. Can you find a transformation such that only a linear least squares problem need be solved?

8. Let A be an m by n matrix and define a diagonal matrix $D = I + \delta \underline{e}_i \underline{e}_i^T$ for $\delta > -1$. Define the LS solution to $\left\| D\left(A\underline{x} - \underline{b}\right)\right\|_2$ by $\underline{x}(\delta)$ and its residual by $\underline{r}(\delta) = \underline{b} - A\underline{x}(\delta)$. Express the k^{th} component of $\underline{r}(\delta)$, denoted by $r_k(\delta)$, in terms of $r_k(0)$.