The University of Connecticut　　　　　　　　　　　　　　　　　Fall 2008
Dept. of ECE　　　　　　　　　　　　　　　　　　　　　　　　KRP

Problem Set # 9

(Due Nov 5, 2008)

(Do only 1, 2,4, 8, 11, 12)

1. A scientist has observed a certain quantity *Q* as a function of a variable *t*. He is interested in determining the mathematical relationship relating *t* and *Q*, which assumes the following polynomial form:

$$Q(t)=\sum_{i=0}^{n-1} a_i t^i;\ a_i \geq 0;\ \sum_{i=0}^{n-1} a_i = 1$$

From the observed results of his n experiments $(t_1,Q_1),\ (t_2,Q_2),\ \ldots,\ (t_m,Q_m),\ m>n$, he wants to determine the coefficients $\{a_i\}$ such that a measure of the error between the observed values and those predicted by the polynomial function is minimized. Consider the following two measures of error:

(i) Minimize $f=\sum_{i=1}^{m} |Q_i - Q(t_i)|$

(ii) Minimize $f=\max_i |Q_i - Q(t_i)|$

Show that the Scientist's problem reduces to linear programming problems **under** both measures of error.

2. Convert the following *LP* problem to standard form:

Maximize $x_1+4x_2+x_3$

Subject to $2x_1-2x_2+x_3=4$

$x_1-x_3=1$

$x_2\geq 0,\ x_3\geq 0$

3. An oil refinery has two sources of crude oil: a light crude that costs \$20 per barrel and a heavy crude that costs \$20 per barrel. The refinery produces gasoline, heating oil, and jet fuel from the crude in the amounts per barrel indicated in the following table:

	Gasoline	Heating Oil	Jet Fuel
Light Crude	0.3	0.2	0.3
Heavy Crude	0.3	0.4	0.2

The refinery has contracted to supply 900,000 barrels of gasoline, 800,000 barrels of heating oil, and 500,000 barrels of jet fuel. The refinery wises to find the amounts of light and heavy crude to buy so as to be able to meet its obligations at minimum cost. Formulate this problem as a linear program, and solve for the optimal solution using: (a) revised simplex method, and (b) Affine scaling version of Karmarkar's algorithm.

4. A small computer manufacturing company forecasts the demand over the next n months to be d_j, i = 1, 2, …, n. In any month, it can produce r units, using **regular** production, at a cost of b dollars per unit. By using **overtime**, it can produce additions units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month. Formulate the problem of determining the production schedule that minimizes cost (Hint: Formulate the problem as a minimization of a piecewise linear function.)

5. Consider an n by n matrix B. Suppose that we want to replace the i^{th} column of B by another n vector $\underline{a}$. That is, we remove the i^{th} column $\underline{b}_i$ from B and put $\underline{a}$ in its place. Write this operation in matrix form.

6. Suppose that we have the LU decomposition of an n by n matrix B. Suppose we replace column i of B (i.e., $\underline{b}_i$) by a new column $\underline{a}$ of the same dimension. Devise an $O(n^2)$ algorithm to find the new L and U.

7. Minimize $7x_2 + 9x_3$ subject to $x_1 + x_2 + x_3 = 6,\ x_2 + 2x_3 + x_4 = 1,\ x_i \geq 0$ using (a) revised simplex method, and (b) Affine scaling version of Karmarkar's algorithm

8. Solve:

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 4x_2 + x_3 + x_4 \\ \text{subject to} \quad & x_1 + 3x_2 + x_4 \leq 4 \\ & 2x_1 + x_2 \leq 3 \\ & x_2 + 4x_3 + x_4 \leq 3 \\ & x_i \geq 0,\ i = 1,\ 2,\ 3,\ 4 \end{aligned}$$

   (a) How much can the elements of $\underline{b}^T = (4,3,3)$ be changed without changing the optimal basis?
   (b) How much can the elements of $\underline{c}^T = (2,4,1,1)$ be changed without changing the optimal basis?
   (c) What happens to the optimal cost for small changes in $\underline{b}$?
   (d) What happens to the optimal cost for small changes in $\underline{c}$?

9. Rather than select the variable corresponding to the most negative relative cost coefficient as the variable to enter the basis, it has been suggested that a better criterion would be to select that variable which, when pivoted in, will produce the greatest improvement in the objective function. Show that this criterion leads to selecting the variable x_{Nk} corresponding to the index k minimizing i: $\max\limits_{a_{ik}>0} \dfrac{P_k \beta_i}{\alpha_{ik}}$

10. Consider an assignment problem of allocating certain activities $i\,(1 \le i \le n)$ to individuals *j* $(1 \le j \le n)$. If activity *i* is allocated to individual *j*, a value of s_{ij} units (dollars) is generated. The objective is to maximize the total value generated, subject to the constraint that at most one activity can be allocated to an individual, and that all activities must be allocated. The problem, also known as the bipartite matching problem, can be formulated as:

$$\max_{\{x_{ij}\}} \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij} x_{ij}$$

subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, 2, ..., n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for } i = 1, 2, ..., n$$

$$x_{ij} = 0,1$$

(a) Write the dual problem formulation.

(b) Show that there exist dual prices $\lambda_i\ (1 \le i \le n)$ and $\mu_i\ (1 \le j \le n)$ such that:

$$\lambda_i + \mu_i \ge s_{ij} \text{ for all } i \text{ and } j$$

with equality holding if in an optimal assignment an activity *i* is assigned to individual *j*.

(c) Show that part (b) implies that if activity *i* is optimally assigned to individual *j* and if *k* is any other individual $s_{ij} - \mu_j \ge s_{ik} - \mu_k$

11. A company manufactures n different products, each of which uses various amounts of m limited resources. Each unit of product yields a profit of c_i dollars and uses a_{ji} units of j^{th} resource. The available amount of the j^{th} resource is b_j. To maximize profit, the company selects the quantities x_i to be manufactured of each product by solving:

$$\max_x c^T x \text{ subject to } Ax \le b,\ x \ge 0$$

The unit profits c_i already take into account the variable cost associated with manufacturing each unit. In addition to that cost, the company incurs a fixed overhead H, and for accounting purposes it wants to allocate this overhead to each of its products. In other words, it wants to adjust c_i^S so as to account for the overhead. Such an overhead allocation scheme must meet two conditions: (1) Since H is fixed regardless of the product mix, the overhead allocation scheme must not alter the optimal solution (i.e., product mix), and (2) all the overhead must be allocated. That is, the optimal value of the objective with the modified cost coefficients must be ***H*** dollars lower than f^* - the original optimal value of the objective function.

(a) Consider the allocation scheme in which the unit profits are modified according to $c_i' = c_i - r\lambda^{*T} a_i$, where λ^* is the optimal dual vector for the original objective and

$r = H / f^*$ (assume $H \le f^*$). Show that the optimal *x* for the modified problem is the same as that of the original problem and the new dual solution is $\lambda^*_{new} = \lambda^*(1-r)$. Show that this approach fully allocates ***H***.

(b) Suppose that the overhead can be traced to each of the resources. Let $H_j \ge 0$ be the amount of overhead associated with the j^{th} resource, where

$$\sum_{j=1}^{m} H_j \le f^*, \text{ and } r_j = \frac{H_j}{b_j} \le \lambda^*_j$$

for j = 1,2, …, m. Based on this information, an allocation scheme has been proposed where the unit profits are modified such that $c_i^{'} = c_i - r^T a_i$. Show that the optimal *x* for this modified problem is the same as that for the original problem, and that the corresponding dual solution is $\lambda^*_{new} = \lambda^* - r$. Show that this scheme also fully allocates H_j.

12. Solve: $\min 2x_1 - x_2$ subject to $2x_1 - x_2 - x_3 \ge 3,\ x_1 - x_2 + x_3 \ge 2,$ and $x_i \ge 0$. (Hint: $x_1 = 2$ is a feasible solution). What is the dual problem and its optimal solution?

13. Game theory is in part related to linear programming theory. Consider a game in which player *X* may select any one of m moves, and player *Y* may select any one of n moves. If *X* selects *i* and *Y* selects *j*, then the player *X* gain a_{ij} units. The players have played the game several times before, so that *X* has developed a mixed strategy where the various moves are played according to probabilities represented by the components of the vector $x = (x_1, x_2, ..., x_m)^T$, $x_i \ge 0$ and the components sum to one. Likewise, *Y* develops a mixed strategy $y = (y_1, y_2, ..., y_n)^T$, $x_j \ge 0$ and the components sum to one. The average value gained by the team of X is then $P(x, y) = x^T A y$

(a) Suppose that *X* select ***x*** as the solution of the linear program:

$$\begin{aligned} &\max\ C \\ &\text{subject to } \sum_{i=1}^{m} x_i = 1 \\ &A^T x \ge C \\ &x_i \ge 0 \end{aligned}$$

Show that *X* is guaranteed a payoff of at least *C* no matter what ***y*** is chosen by *Y*.

(b) Show that the dual of the problem above is:

$$\min\ B$$

$$\text{subject to } \sum_{j=1}^{n} y_j = 1$$

$$Ay \le B$$

$$y_j \ge 0$$

(c) Prove that $\max C = \min B$

(d) Consider the “matching game”. Each player select heads or tails. If the choices match, *X* win \$1 from *Y*; if they do not match, *Y* win \$1 from *X*. Find the value of this game and the optimal mixed strategies.

(e) Repeat part (d) for the game where each player selects either 1, 2, or 3. The player with the highest number win \$1 unless the number is exactly 1 higher than the other player’s number, in which case he loses \$3. When the numbers are equal, there is no pay off.

(f) Repeat part (d) for a game in which *m* = *n* = 2, and $a_{11} = 1,\ a_{21} = 3,\ a_{12} = 4,$ and $a_{22} = 2,$