

Outline of Lecture 3

- \Box What is the need for computing $\int e^{As} ds$, etc.?
- \Box How to get integrals from the exponential of a modified matrix?
- Concept of **doubling**
- Error analysis
- **Application to system stabilization**

References

- 1. C. F. Van Loan., "Computing integrals involving the matrix exponential," IEEE Trans. on AC, Vol. AC-23 No-3, June 1978, pp. 395-404.
- 2. E. S. Armstrong, "Series representations for the weighting matrices in sampled-data optimal linear regulator problem," IEEE Trans. on AC, Vol. AC-23, No-3, June 1978, pp. 478-479.
- 3. K. R. Pattipati and S. A. Shah, "On the computational aspects of the performability models of fault-tolerant computer systems," IEEE Trans. on Computers, Vol. C-39, No. 6, June-1990, pp. 832-836.

 \Box What is the need for computing $\int e^{As} ds$, etc.?:

1) $\underline{\dot{x}}(t) = A\underline{x}(t) + Bu(t)$ if $\underline{u}(t)$ is piece-wise –constant over $[k\delta, (k+1) \delta] \forall k$ $(k+1) = e^{A\sigma} \underline{x}(k) + | \underline{e}^{A\sigma} B d\sigma | \underline{u}(k)$ $\underline{x}(k+1) = e^{A\delta}\underline{x}(k) + \int_0^{\delta} e^{A\sigma} B d\sigma \, \mu(k)$ δ if $\underline{u}(t)$ is piece-wise -const.
 $\delta_{x}(k) + \left[\int_{-\infty}^{\delta} e^{A\sigma} B d\sigma \right] u(k)$ \Rightarrow $\underline{x}(k+1) = e^{A\delta} \underline{x}(k) + \left[\int_0^{\delta} e^{A\sigma} B d\sigma \right] \underline{u}(k)$

 $x(k) = x(k\delta)$, $u(k) =$ value of $u(t)$ in the interval $[k\delta, (k+1)\delta)$. **Arises when we discretize a continuous LTI system**

2) Covariance analysis of stochastic LTI systems $f(t) = A \underline{x}(t) + E \underline{w}(t);$ $\underline{X}_C(t) = E \left\{ (\underline{x}(t) - \overline{\underline{x}}(t)) (\underline{x}(t) - \overline{\underline{x}}(t))^T \right\}$ $(k+1) = e^{A\phi} X_C(k)$ $\delta(\delta) = \int_{0}^{\infty} e^{A\sigma} EWE^{T} e^{A^{T}\sigma} d\sigma$ PD if $(A, EW^{1/2})$ 0 0 1 ${}^{T\sigma}d\sigma$ PD if $(A, EW^{1/2})$ controllable $A\delta_{X_C}(k)e^{A^T\delta} + \int^{\delta} e^{A\sigma} EWE^T e^{A^T\delta}$ Covariance analysis of stochastic LTI systems
 $\dot{x}(t) = A \underline{x}(t) + E \underline{w}(t);$ $\underline{X}_C(t) = E \{ (\underline{x}(t) - \overline{\underline{x}}(t)) (\underline{x}(t) - \overline{\underline{x}}(t))^T \}$ $\underline{X}(t) = A\underline{X}(t) + E\underline{W}(t);$ $\Delta_C(t) = E\left\{(\underline{X}(t) - \underline{X}(t))\right\}$
 $\underline{X}_C(k+1) = e^{A\delta}\underline{X}_C(k)e^{A^T\delta} + \int_0^\delta e^{A\sigma}EWE^Te^{A^T\sigma}d$ $\begin{array}{c} - \infty \wedge \ \theta \ \end{array}$ $\Delta_C (k+1) = e \Delta_C (k) e + \int_0^b e^{K} E W E e$
 $S(\delta) = \int_0^{\delta} e^{A \sigma} E W E^T e^{A^T \sigma} d\sigma$ PD if $(A, E W)$ δ $\delta X_c(k)e^{A^T\delta} + \int_0^{\delta} e^{A\sigma} EWE^T e^{A^T\sigma} d\sigma$ δ $\sigma_{EWE}^T e^{A^T \sigma} d\sigma$ $\sum_{C} (k+1) = e^{-\alpha} \Delta_C (k) e^{-\alpha} + \int_0^{\infty} e^{-\alpha}$
 δ) = $\int_0^{\delta} e^{A\sigma} EWE^T e^{A^T\sigma} d\sigma$ PD $A\underline{x}(t) + E\underline{w}(t);$ $\Delta_C(t) = E\{t$
+1) = $e^{A\delta} \underline{X}_C(k) e^{A^T\delta} + \int_0^{\delta} e^{A\sigma} E$ $=\int$ \dot{x}

Why Compute Integrals of
$$
e^{At}
$$
? - 2

• Suppose want to model the continuous stochastic LTI system by its discrete counterpart

 $x(k+1) = \Phi x(k) + \Gamma w(k)$

- Find Φ , Γ and $cov[\underline{w}(k)] = W_d$ $\Rightarrow \underline{X}_d(k\delta) = \underline{X}_c(k\delta)$ at the sampled points, where $X_d(k+1) = \Phi X_d(k) \Phi^T + \Gamma W_d \Gamma^T$
- Two possibilities: (1) $\Phi = e^{A\delta}$, $\Gamma = I$, $W_d = S(\delta)$ (2) $\Phi = e^{A\delta}$, $\Gamma = |S(\delta)|$ $\Phi = e^{A\delta}, \ \Gamma = I, \ \ W_d = S(\delta)$ $^{A\delta}$, $\Gamma = [S(\delta)]^{1/2}$, $\Phi = e^{A\delta}, \ \Gamma = [S(\delta)]^{1/2}, \ W_d = I$
- We will see in Lecture 5 how to compute *square roots of positive definite matrices*
- 3) Integrals of the form $\int_{a}^{t_f} e^{A\sigma}BB^T e^{A^T\sigma} d\sigma$ are used to test the controllability of LTI systems and to solve the minimum energy control problem: \int $(\underline{u}^T \underline{u})dt$ s.t. $\underline{\dot{x}} = A\underline{x} + B\underline{u}$ and $\underline{x}(t_f)$ 0 1 systems and to solve the minimum energy control p
 $\min \frac{1}{2} \int_0^{t_f} \left(\underline{u}^T \underline{u} \right) dt$ s.t. $\underline{x} = A \underline{x} + B \underline{u}$ and $\underline{x} (t_f) = 0$ *f t T* and to solve the minimum energy control probl
 $\int_0^{t_f} \left(\underline{u}^T \underline{u} \right) dt$ s.t. $\underline{x} = A \underline{x} + B \underline{u}$ and $\underline{x} (t_f) = 0$ \dot{x}

a a
Sid

Why Compute Integrals of *e At* **? - 3**

Hamiltonian:
$$
\frac{1}{2} \underline{u}^T \underline{u} + \lambda^T (A \underline{x} + B \underline{u})
$$

\n
$$
\frac{\partial H}{\partial \underline{u}} = 0 \implies \underline{u} + B^T \underline{\lambda} = 0 \implies \underline{u} = -B^T \underline{\lambda}
$$

\n
$$
\frac{\partial H}{\partial \underline{x}} = -\underline{\lambda} = A^T \underline{\lambda}; \ \underline{\lambda} (t_f) = \underline{v} \ (\underline{v} \text{ unknown}) \implies \underline{\lambda} (t) = e^{A^T (t_f - t)} \underline{v}
$$

\n
$$
\underline{u} = -B^T e^{A^T (t_f - t)} \underline{v} \implies x(t_f) = e^{At_f} \underline{x}_0 - \int_0^{t_f} e^{At} BB^T e^{A^T t} dt \underline{v}
$$

\n
$$
\implies \underline{v} = \left[\int_0^{t_f} e^{At} BB^T e^{A^T t} dt \right]^{-1} e^{At_f} \underline{x}_0
$$

\n
$$
\underline{u}(t) = -B^T e^{-A^T t} e^{A^T t_f} \left[\int_0^{t_f} e^{At} BB^T e^{A^T t} dt \right]^{-1} e^{At_f} \underline{x}_0
$$

4) Integrals of the form $\int_{0}^{t} e^{A^T \sigma} C^T C e^{A \sigma} d\sigma$ also rise in testing the observability of LTI systems. 0 $f^{\prime}{}_{f}e^{A^{T}\sigma}C^{T}Ce^{A\sigma}d\sigma$ \int

Why Compute Integrals of
$$
e^{At}
$$
? - 4

5) In sampled data regulator problem, in addition to $\Gamma(\delta)$ and $S(\delta)$, we appled data regulator problem, in addition to $\Gamma(\delta)$ and $S(\delta)$, we set integrals of the form:
 δ) = $\int_0^{\delta} e^{A^T \sigma} Q \Gamma(\sigma) d\sigma$ and $N(\delta) = \int_0^{\delta} \Gamma^T(\sigma) Q \Gamma(\sigma) d\sigma$

also get integrals of the form:
\n
$$
M(\delta) = \int_0^{\delta} e^{A^T \sigma} Q \Gamma(\sigma) d\sigma \text{ and } N(\delta) = \int_0^{\delta} \Gamma^T(\sigma) Q \Gamma(\sigma) d\sigma
$$

6) Performability models of fault-tolerant computer systems
 $\overline{S} = \epsilon^T \left[\int_{-\infty}^T e^{Q^T t} dt \right]$

$$
\mathbf{y}_{[0\ T]} = \underline{f}^T \left[\int_0^T e^{\mathcal{Q}^T t} dt \right] \underline{p}_0
$$

• So, need:

$$
\begin{aligned} \n\text{where} \quad \mathbf{C} \cdot \mathbf{D} &= \int_0^\delta e^{A\sigma} B d\sigma \\ \n&= S(\delta) = \int_0^\delta e^{A\sigma} B B^T e^{A^T \sigma} d\sigma \text{ or } \int_0^\delta e^{A^T \sigma} C^T C e^{A\sigma} d\sigma \n\end{aligned}
$$

Computation of Γ **and** $\Phi - 1$

Computation of Γ and Φ : Method A

- Here δ is usually small w.r.t $1/||A||$, typically 0.1/ $||A||$ or 0.2 / $||A||$
- 0 $e^{A\sigma}$ *d* δ $\Psi = \int_0^b e^{A\sigma} d\sigma$
-

• So, Taylor series for
$$
e^{A\sigma}
$$
 is good, since $||A\delta|| < 1$
\n
$$
\Psi = \int_0^{\delta} \left(I + A\sigma + \frac{A^2 \sigma^2}{2!} + \dots \right) d\sigma = \delta I + \frac{A\delta^2}{2} + \frac{A^2 \delta^3}{3!} + \dots + \frac{A^k \delta^{k+1}}{(k+1)!}
$$

- Then, $\Gamma = \Psi B$, $\Phi = I + A \Psi$
- Input *k* to routine, $k \approx 4$, $||A\delta|| = 0.1 \Rightarrow$ error $\approx 10^{-4}\delta/120 \approx 10^{-6}\delta$
- Function $c2d$ in MATLAB computes Φ and Γ
- **Widely used method in digital control**

Computation of Γ **and** Φ **– 2**

Computation of Γ and Φ : Method B

- $de^{At} / dt = Ae^{At}$; $e^{At} |_{t=0} = I$ or $d\Phi / dt = A\Phi(t)$; $\Phi(0)$ ˆ $\begin{bmatrix} 1 & \Gamma \end{bmatrix}$, $\frac{d}{d\epsilon} \begin{bmatrix} e^{Ct} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \Phi_1 \end{bmatrix}$ $\hat{C} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}; \quad \underline{e}^{\hat{C}t} = \begin{bmatrix} \Psi_1 & 1 \\ 0 & \Phi_2 \end{bmatrix}; \quad \frac{d}{dt} \begin{bmatrix} e^{Ct} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi_1 & 1 \\ 0 & \Phi_2 \end{bmatrix}$ $\mathbf{I}_1 = A \Phi_1 \implies \Phi_1 = e^{At}; \quad \dot{\Phi}_2 = 0 \implies \Phi_2$ 2 0 tion of Γ and Φ : Method B
 $/dt = Ae^{At}$; $e^{At} |_{t=0} = I$ or $d\Phi/dt = A\Phi(t)$; $\Phi(0)$ $d\theta = Ae^{At}$; $e^{At} |_{t=0} = I$ or $d\Phi/dt = A\Phi(t)$; $\Phi(C)$
 $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$; $e^{\hat{C}t} = \begin{bmatrix} \Phi_1 & \Gamma \\ 0 & \Phi_2 \end{bmatrix}$; $\frac{d}{dt} \begin{bmatrix} e^{Ct} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}$ \Rightarrow $\Phi_1 = e^{At}$; $\dot{\Phi}_2 = 0$ $\Rightarrow \dot{\Gamma} = A\Gamma + B \Rightarrow \Gamma = \left| \int_{0}^{b} e^{A} \right|$ putation of Γ and Φ : Method B
 $de^{At} / dt = Ae^{At}$; $e^{At} |_{t=0} = I$ or $d\Phi / dt = A\Phi(t)$; $\Phi(0) = I$ $de^{At}/dt = Ae^{At}$; $e^{At} |_{t=0} = I$ or $d\Phi/dt$
 $\hat{C} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$; $e^{\hat{C}t} = \begin{bmatrix} \Phi_1 & \Gamma \\ 0 & \Phi_2 \end{bmatrix}$; $\frac{d}{dt} \begin{bmatrix} e^{At} \\ e^{At} \end{bmatrix}$ 0 0 $\int_0^1 \frac{e^{2t}}{t} dt = \int_0^1 0 \Phi_2 \int_0^1 dt \left[\frac{e^{2t}}{t} \right] dt$
 A $\Phi_1 \Rightarrow \Phi_1 = e^{At}$; $\dot{\Phi}_2 = 0 \Rightarrow \Phi_2 = I$ $\Phi_1 \Rightarrow \Phi_1 = e^{At}; \quad \Phi_2 = 0 \Rightarrow \Phi_2 = I$
 $A\Gamma + B\Phi_2 \Rightarrow \dot{\Gamma} = A\Gamma + B \Rightarrow \Gamma = \left[\int_0^{\delta} e^{A\sigma} d\sigma\right]B$ δ ${}^{\sigma}d\sigma$ $/dt = Ae^{At}$; $e^{At}|_{t=0} = I$ or $d\Phi/dt = A\Phi(t)$; $\Phi(0) = I$
 $\begin{bmatrix} A & B \end{bmatrix}$; $e^{\hat{C}t} = \begin{bmatrix} \Phi_1 & \Gamma \end{bmatrix}$; $\frac{d}{d\Phi}e^{Ct} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \Phi_1 & \Gamma \end{bmatrix}$ $A^t / dt = Ae^{At}$; $e^{At} |_{t=0} = I$ or $d\Phi / dt = A\Phi(t)$; $\Phi(0) = I$
= $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$; $e^{\hat{C}t} = \begin{bmatrix} \Phi_1 & \Gamma \\ 0 & \Phi_2 \end{bmatrix}$; $\frac{d}{dt} \begin{bmatrix} e^{Ct} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_1 & \Gamma \\ 0 & \Phi_2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 0 \ 0 & 0 & 0 \end{bmatrix}$; $e^{Ct} = \begin{bmatrix} 1 & 0 \ 0 & \Phi_2 \end{bmatrix}$; $\frac{\alpha}{dt} \begin{bmatrix} e^{Ct} \end{bmatrix} = \begin{bmatrix} 1 & 0 \ 0 & 0 \end{bmatrix}$
 $\dot{\Phi}_1 = A\Phi_1 \implies \Phi_1 = e^{At}$; $\dot{\Phi}_2 = 0 \implies \Phi_2 = I$ $\dot{\Phi}_1 = A\Phi_1 \implies \Phi_1 = e^{At}; \quad \dot{\Phi}_2 = 0 \implies \Phi_2 = I$
 $\dot{\Gamma} = A\Gamma + B\Phi_2 \implies \dot{\Gamma} = A\Gamma + B \implies \Gamma = \left[\int_0^{\delta} e^{A\sigma} d\sigma\right]B$
- If we want $\Gamma(\delta)$, all we need to do is to find $e^{\hat{C}\delta}$ and take $\Gamma(\delta)$ as the $(1,2)$ block of $e^{\hat{C}\delta}$.
- *Note:* δ need not be small with this approach, since we can use inc
:on
n_ix

shifting, scaling, and doubling techniques to compute
$$
e^{\hat{C}\delta}
$$
.
\n
$$
e^{\hat{C}\delta} = \left[\hat{f}_{pade}\left(\frac{\hat{C}\delta}{2^M}\right)\right]^{2^M} \text{ or } \left[\hat{f}_{chebyshev}\left(\frac{\hat{C}\delta}{2^M}\right)\right]^{2^M}; \hat{f}_{pade}(x) = \frac{\sum_{i=0}^m n_i x^i}{\sum_{i=0}^m (-1)^i n_i x^i}, n_i = \frac{(2m-i)!}{2m!} {m \choose i}
$$

Doubling Equations for Γ **and** Φ

- Approach
	- $-\text{Find } M \supset \left\| \hat{C} \delta / 2^M \right\| \leq 1/2$ $-$ Let $\Delta = \delta^{\parallel}/2^M$
	- $-$ Find $\Phi_1(\Delta)$ and $\Gamma(\Delta)$ 2^M $\begin{bmatrix} 2^M & 2 \end{bmatrix}$ and $\Gamma(\Delta)$ by PADE or Chebyshev $e^{A\Delta}$, $\left[\int_0^{\Delta} e^{A\sigma} d\sigma \right]$ *B* M $\ni |\tilde{C}\delta/2^M| \le 1/2$
 $\Delta = \delta/2^M$
 $\Phi_1(\Delta)$ and $\Gamma(\Delta)$ by PADE or Chebyshev $e^{A\Delta}$, $\left[\int_0^{\Delta} e^{A\sigma} d\sigma\right]B$

	use the fact that: $e^{\hat{C}2t} = e^{\hat{C}t}e^{\hat{C}t}$ - Then use the fact that: $e^{\hat{C}2t} = e^{\hat{C}t}e^{\hat{C}t}$

$$
,\ \bigg[\int_0^\Delta e^{A\sigma}d\sigma\bigg]B
$$

• *Note:* don't need to carry all the elements
\n
$$
\Rightarrow \begin{bmatrix} \Phi_1(2t) & \Gamma(2t) \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi_1(t) & \Gamma(t) \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_1(t) & \Gamma(t) \\ 0 & I \end{bmatrix}
$$
\n
$$
= \begin{bmatrix} \Phi_1^2(t) & \Gamma(t) + \Phi_1(t) \Gamma(t) \\ 0 & I \end{bmatrix}
$$

In place computation:

In place computation:
\n
$$
\Gamma(2t) = \Gamma(t) + \Phi_1(t)\Gamma(t) = [I + \Phi_1(t)]\Gamma(t)
$$
\n
$$
\Phi_1(2t) = \Phi_1^2(t)
$$

Practicalities

• Don't actually need to evaluate \hat{C}^k as an $(n+m)$ by $(n+m)$ matrix in

using Pade approximation. Some simplifications are possible!!
\n
$$
\hat{C} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}; \quad \hat{C}^2 = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^2 & AB \\ 0 & 0 \end{bmatrix}
$$
\n
$$
\hat{C}^k = \begin{bmatrix} X_k & Y_k \\ 0 & 0 \end{bmatrix} \implies \begin{aligned} X_k &= AX_{k-1}, X_0 = I \\ Y_k &= AY_{k-1}, Y_0 = B \end{aligned}
$$

 \Rightarrow need one *n* x *n* and one *n* x *m* matrix

• **2** each of
$$
n \times n
$$
 and one $n \times m$ matrix

\n**PADE**
$$
e^{\hat{C}\Delta} = \left[\sum_{k=0}^{m} n_k \hat{C}^k \left(-\Delta \right)^k \right]^{-1} \left[\sum_{k=0}^{m} n_k \hat{C}^k \Delta^k \right]
$$

• Use Horner's rule
$$
\Rightarrow 3m
$$
 matrix multiples
\n
$$
\hat{D} = \begin{bmatrix} D_{11} & D_{12} \\ 0 & I \end{bmatrix}; \quad \hat{N} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & I \end{bmatrix}; \quad \begin{bmatrix} D_{11} & D_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & I \end{bmatrix}
$$
\n
$$
\Rightarrow \text{solve} \quad D_{11}\Phi_1(\Delta) = N_{11} \Rightarrow D_{11}(\Phi_1...\Phi_n) = (\underline{n}_1...\underline{n}_n)_{11}
$$
\n
$$
D_{11}\Gamma(\Delta) + D_{12} = N_{12} \Rightarrow D_{11}(\Gamma_1...\Gamma_m) = (\underline{n}_1 - \underline{d}_1...\underline{n}_m - \underline{d}_m)
$$

Error Analysis - 1

• <u>Same</u> D_{11} matrix.. Can exploit this observation using the LU decomposition techniques of Lecture 4 for solving *Ax*=*b*. What is the error made? (see Golub and Van Loan)

 $\Vert e^{-\alpha} \Vert \leq \varepsilon \delta \theta(\delta)$ $(\delta)e^{i\theta}|1+\alpha\delta/2|$ (2^{5-2m}) ||C||(m!) $(2m+1)$ 0 ^{3-2*m*} $\|\hat{C}\|$ (*m*!)² $\left[1 + \alpha \delta / 2\right]$ $\frac{\mathbb{I}}{2^{3-2m}}\Big\|\hat{C}\Big\|(m!)$ where $\varepsilon = \frac{y - \mu}{2}$, $\epsilon \ge ||E_A||$, E_A = error in $\frac{2m}{2m!(2m+1)!}$ *A A m* $A\parallel$, E_A t is the error made? (see
 $\Phi_1 - e^{A\delta}$ $\leq \varepsilon \delta \theta(\delta) e^{\varepsilon \delta}$ $e^{A\sigma}Bd\sigma \leq \varepsilon \delta\theta(\delta)e$ \hat{C} \parallel (m) E_A ^{\parallel}, E_A = error in A $\frac{2m}{m!(2m)}$ δ $\Gamma - \int_0^{\delta} e^{A\sigma} B d\sigma \leq \varepsilon \delta \theta(\delta) e^{\varepsilon \delta} \left[1 + \alpha \delta / 2 \right]$ $\varepsilon = \frac{(2^{3-2m}) ||\hat{C}|| (m!)^2}{2m!(2m+1)!}, \ \varepsilon \ge ||E_A||, \ E_A = \text{error}$ $\ddot{}$ (δ) 0 max $||e^{As}||$; *s* e^{As} ||; $\alpha =$ ||*B* δ $\theta(\delta) = \max ||e^{As}||$; $\alpha =$ $\frac{\text{max}}{\leq s \leq \delta}$ $=$ max $||e^{As}||$; $\alpha = ||E||$ \Rightarrow can control the accuracy of the algorithm via *m*. e.g., choose *m* to satisfy

can control the accuracy of the algorithm via *m*. e.g., choose *m* to satisfy\n
$$
\left\|\Phi_1 - e^{A\delta}\right\| \le \varepsilon \delta e^{\varepsilon \delta} \le \text{TOL and } \frac{\left\|\Gamma - \int_0^\delta e^{A\sigma} B d\sigma\right\|}{\left\|e^{A\delta}\right\|} \le \varepsilon \delta e^{\varepsilon \delta} \left[1 + \alpha \delta / 2\right] \le \text{TOL}
$$

Error Analysis - 2

Proof:
$$
\|\Phi_1 - e^{A\delta}\| = \|e^{(A+E_A)\delta} - e^{A\delta}\| \le \|e^{A\delta}\| \|e^{E_A\delta} - I\| \le \theta(\delta)\varepsilon \delta e^{\varepsilon \delta}
$$

see Moler and Van loan for a proof that *A* and *E^A* commute

$$
\frac{\text{Proof: } \|\Phi_1 - e^{\infty}\| = \|e^{(\Lambda + \Sigma_A)\sigma} - e^{\infty}\| \le \|e^{\infty}\| \|e^{\Sigma_A \sigma} - I\| \le \theta(\delta)\epsilon\delta e^{\infty}
$$
\n
$$
\text{see Moler and Van loan for a proof that } A \text{ and } E_A \text{ commute}
$$
\n
$$
\Gamma - \int_0^\delta e^{A\sigma} B d\sigma = \left[\int_0^\delta e^{(A + E_A)\sigma} (B + E_B) d\sigma - \int_0^\delta e^{A\sigma} B d\sigma \right]
$$
\n
$$
= \int_0^\delta e^{(A + E_A)\sigma} B d\sigma - \int_0^\delta e^{A\sigma} B d\sigma + \int_0^\delta e^{(A + E_A)\sigma} E_B d\sigma
$$
\n
$$
= \int_0^\delta \left[e^{(A + E_A)\sigma} - e^{A\sigma} \right] B d\sigma + \int_0^\delta e^{(A + E_A)\sigma} E_B d\sigma
$$
\n
$$
\le \int_0^\delta \|e^{A\sigma}\| \epsilon\sigma e^{\epsilon\sigma} \|B\| d\sigma + \int_0^\delta \|e^{A\sigma}\| \epsilon e^{\epsilon\delta} d\sigma
$$
\n
$$
\le \int_0^\delta \theta(\delta) \epsilon e^{\epsilon\delta} \alpha \sigma d\sigma + \theta(\delta) \epsilon e^{\epsilon\delta} \delta
$$
\n
$$
= \epsilon\delta\theta(\delta) e^{\epsilon\delta} \left[1 + \frac{\alpha\delta}{2} \right]
$$

Grammian Type Integrals - 1

- Closed form solution for idempotent matrices:
- *A* is idempotent $\Rightarrow \Psi = \delta(I-A) + A(e^{\delta}-1); \Phi = I + A(e^{\delta}-1)$
- Use as test cases.

Q Computation of $S = \int_{0}^{b} e^{A\sigma} Q e^{A'\sigma} d\sigma$, $Q = Q^T$ PSD $\Rightarrow \lambda_i(Q)$ $e^{A\sigma} Q e^{A^T \sigma} d\sigma$, $Q = Q^T$ PSD $\Rightarrow \lambda_i(Q) \ge 0$, or $\underline{x}^T Q \underline{x} \ge 0$ *T* Ses.
 $S = \int_0^{\delta} e^{A \sigma} Q e^{A^T \sigma} d\sigma$, $Q = Q^T$ PSD $\Rightarrow \lambda_i(Q) \ge 0$, or $\underline{x}^T Q \underline{x} \ge 0 \forall \underline{x}$ δ S.
= $\int_0^{\delta} e^{A\sigma} Q e^{A^T \sigma} d\sigma$, $Q = Q^T$ PSD $\Rightarrow \lambda_i(Q) \ge 0$, or $\underline{x}^T Q \underline{x} \ge 0 \ \forall \underline{x}$

• All schemes use doubling concept, i.e., first find \int_0 $e^{A\sigma}$ Q $e^{A^T\sigma}$ d σ \int_0^Δ

where ||A∆|| is small (≤1/2) and then get $\int_{c}^{b} e^{A\sigma} Q e^{A^T\sigma} d\sigma$ by doubling $\int_{0}^{\delta} e^{A \sigma} Q e^{A^T \sigma} d\sigma$

where
$$
||A\Delta||
$$
 is small $(\leq 1/2)$ and then get $\int_0^b e^{A\sigma} Q e^{A^T \sigma} d\sigma$
\n• Consider $\hat{C} = \begin{bmatrix} -A & Q \\ 0 & A^T \end{bmatrix}$; $e^{\hat{C}t} = \begin{bmatrix} \Phi_1 & G \\ 0 & \Phi_2 \end{bmatrix}$

$$
\begin{array}{cccc}\n & 0 & A^I \end{array} \implies \Phi_1 = e^{-At};
$$
\n
$$
\dot{G} = -AG + Q\Phi_2; \quad G(0) = 0
$$

$$
\dot{\Phi}_2 = A^T \Phi_2 \Rightarrow \Phi_2(t) = e^{A^T t} = \left[\Phi_1^{-1}(t)\right]^T
$$

$$
\dot{\Phi}_2 = A^T \Phi_2 \Rightarrow \Phi_2(t) = e^{A^T t} = \left[\Phi_1^{-1}(t)\right]^T
$$

\n
$$
\Rightarrow G(t) = \int_0^t e^{-A(t-\sigma)} Q e^{A^T \sigma} d\sigma = e^{-At} \int_0^t e^{A\sigma} Q e^{A^T \sigma} d\sigma
$$

Error Analysis

 $S(\Delta) = \Phi_2^T(\Delta) G(\Delta)$ MUST MAKE THIS SYMMETRIC

$$
S(\Delta) = \frac{1}{2} \left[\Phi_2^T(\Delta) G(\Delta) + G^T(\Delta) \Phi_2(\Delta) \right]
$$

- $\Gamma_1(\Delta)$ and $S(\Delta)$ from exp $\{C\Delta\}$ \Rightarrow Compute $\Phi_1(\Delta)$ and $S(\Delta)$ from $\exp{\{\hat{C}\Delta\}}$
- PADE or Chebyshev approximation \Rightarrow
-

S(
$$
\Delta
$$
) = $\Phi_2^L(\Delta)G(\Delta)$ MUST MAKE THIS SYMMETRIC
\n
$$
S(\Delta) = \frac{1}{2} [\Phi_2^T(\Delta)G(\Delta) + G^T(\Delta)\Phi_2(\Delta)]
$$
\n
$$
\Rightarrow \text{ Compute } \Phi_1(\Delta) \text{ and } S(\Delta) \text{ from } \exp{\hat{C}\Delta}
$$
\n
$$
\Rightarrow \text{ PADE or Chebyshev approximation}
$$
\n
$$
\Box \text{ What is the error made?}
$$
\n
$$
|\Phi(\Delta) - e^{A\delta}| \le \varepsilon \Delta \theta(\Delta) e^{\varepsilon \delta} \text{ same as earlier}
$$
\n
$$
|S(\Delta) - \int_0^{\Delta} e^{A\sigma} Q e^{A^T \sigma} d\sigma| \le \varepsilon \Delta \theta^2(\Delta) e^{2\varepsilon \Delta} (1 + \alpha_q \Delta), \quad \alpha_q = ||Q||
$$
\nProof: $S(\Delta) - \int_0^{\Delta} e^{A\sigma} Q e^{A^T \sigma} d\sigma = \int_0^{\Delta} e^{(A + E_A)\sigma} (Q + E_Q) e^{(A + E_A)^T \sigma} d\sigma - \int_0^{\Delta} e^{A\sigma} Q e^{A^T \sigma} d\sigma$ \n
$$
= \int_0^{\Delta} [e^{(A + E_A)\sigma} - e^{A\sigma}] Q[e^{(A + E_A)^T \sigma} - e^{A^T \sigma}] d\sigma
$$
\n
$$
+ \int_0^{\Delta} e^{A\sigma} Q[e^{(A + E_A)^T \sigma} - e^{A^T \sigma}] d\sigma + \int_0^{\Delta} e^{(A + E_A) \sigma} E_Q e^{(A + E_A)^T \sigma} d\sigma
$$

 $2\varepsilon\Delta_{\alpha} \theta^2(\Delta)d\sigma + \int^{\Delta} \varepsilon \sigma \theta^2(\Delta) e^{\varepsilon \Delta} d\sigma + \int^{\Delta} \theta^2(\Delta) \varepsilon e^2$)
 $^{2}(\Delta)\varepsilon e^{2}$ $\left[2\frac{2}{(\Delta)\varepsilon e^{2\varepsilon\Delta}}\right] \Delta + \alpha_q \Delta^2$ $\int_0^{\Delta} \varepsilon \sigma e^{2\varepsilon \Delta} \alpha_q \theta^2(\Delta) d\sigma + \int_0^{\Delta} \varepsilon \sigma \theta^2(\Delta) e^{\varepsilon \Delta} d\sigma + \int_0^{\Delta}$ Take norms $\leq \int_0^{\Delta} \varepsilon \sigma e^{2\varepsilon \Delta} \alpha_q \theta^2(\Delta) d\sigma + \int_0^{\Delta} \varepsilon \sigma \theta^2(\Delta) e^{\varepsilon \Delta} d\sigma + \int_0^{\Delta} \theta^2(\Delta) \varepsilon e^{2\varepsilon \Delta} d\sigma$ $\frac{d^2}{2} + e^{-\varepsilon \Delta} \frac{\alpha_q}{2}$ *q q e e* $\begin{split} \Delta \big) d\sigma + \int_0^{\infty} &\varepsilon \sigma \theta^2 \big(\Delta \big) e^{\varepsilon \Delta} d\sigma \ \frac{\alpha_q \Delta^2}{1 + e^{-\varepsilon \Delta}} &\frac{\alpha_q \Delta^2}{1 + e^{-\varepsilon \Delta}} \bigg] \end{split}$ θ (Δ)εe Δ + $\frac{\Delta}{2}$ + e
 θ ²(Δ)εe^{2εΔ} $\left[\Delta + \alpha_q \Delta^2\right]$ $\leq \int_0^{\Delta} \varepsilon \sigma e^{2\varepsilon \Delta} \alpha_q \theta^2(\Delta) d\sigma + \int_0^{\Delta} \varepsilon \sigma \theta^2(\Delta) e^{\varepsilon \Delta} d\sigma + \int_0^{\Delta} \theta^2(\Delta) \varepsilon e^{2\varepsilon \Delta} d\sigma$ I_0 $\cos \alpha_q \cos \beta_q$
= $\theta^2(\Delta) \varepsilon e^{2\varepsilon \Delta} \left[\Delta + \frac{\alpha_q \Delta^2}{2} + e^{-\varepsilon \Delta} \frac{\alpha_q \Delta^2}{2} \right]$ $\theta^2(\Delta)d\sigma + \int_0^{\Delta} \varepsilon \sigma \theta^2 (\Delta) e^{\varepsilon \Delta} d\sigma + \int_0^{\Delta} \left(\Delta + \frac{\alpha_q \Delta^2}{4} + e^{-\varepsilon \Delta} \frac{\alpha_q \Delta^2}{4} \right)$ $\left[\Delta + \frac{\alpha_q \Delta^2}{2} + e^{-\varepsilon \Delta} \frac{\alpha_q \Delta^2}{2}\right]$ = $\theta^2(\Delta) \varepsilon e^{2\varepsilon \Delta} \left[\Delta + \frac{q}{2} + e^{-\varepsilon \Delta} \frac{q}{2} \right]$
 $\leq \theta^2(\Delta) \varepsilon e^{2\varepsilon \Delta} \left[\Delta + \alpha_q \Delta^2 \right]$ **Practicalities - 1**
 $\int_0^{\Delta} \varepsilon \sigma e^{2\varepsilon \Delta} \alpha_q \theta^2(\Delta) d\sigma + \int_0^{\Delta} \varepsilon \sigma \theta^2(\Delta) e^{\varepsilon \Delta} d\sigma + \int_0^{\Delta} \theta^2(\Delta) \varepsilon e^{2\varepsilon \Delta} d\sigma$ **Q** Computation of $e^{\hat{C}\Delta}$ can be simplified $e^{\hat{C}\Delta}$ $(-1)^k X_k$ R_k $X_k = AX_{k-1}$ $1 + QX_{k-1}^T$ $\hat{C}^k = \begin{bmatrix} (-1)^k X_k & R_k \end{bmatrix}$ 0 *k* $X_k = \left[(-1)^k X_k \quad R_k \right] \implies X_k = AX_k$ $\begin{bmatrix} T \\ K \end{bmatrix} \Rightarrow R_k = -AR_{k-1} + QX_k^T$ $\hat{C}^k = \begin{bmatrix} (-1)^k X_k & R_k \end{bmatrix} \implies X_k = AX$ $\begin{array}{c} R_k \ X_k^T \end{array} \Rightarrow \begin{array}{c} X_k = A X_{k-1} \ R_k = -A R_{k-1} + Q X_k \end{array}$ $-1 + QX_{k-1}^T$ ation of e can be simplified
 $\begin{bmatrix} (-1)^k X_k & R_k \end{bmatrix}$ $X_k =$ $=\left[\begin{pmatrix} -1 \end{pmatrix}^k X_k \quad R_k \right] \Rightarrow$ $\begin{bmatrix} (-1)^k X_k & R_k \\ 0 & X_k^T \end{bmatrix} \Rightarrow \begin{aligned} X_k &= AX_{k-1} \\ R_k &= -AR_{k-1} + QX_k^T. \end{aligned}$ • Need 2 *n* x *n* matrices *X* and *R* • Compute $N = \sum n_i \overline{C}^i \Delta^i$; $\overline{D} = \sum n_i (-1)$ $\begin{array}{c}\n0 \end{array}$ $\begin{array}{c}\n\hline\n0 \end{array}$ $\begin{array}{c}\n\hline\n1 \end{array}$ $\hat{V} = \sum_{i=1}^{m} n_i \hat{C}^i \Delta^i; \quad \hat{D} = \sum_{i=1}^{m} n_i (-1)^i \hat{C}^i \Delta^i$ natrices X and R $\frac{m}{M}$ $i \wedge i$. $\hat{D} = \sum^{m} n \cdot (-1)^i \hat{C}^i \wedge i$ $i\hat{C}^i\Delta^i; \hspace{12pt} \hat{D} = \sum n_i$ $\sum_{i=0}^{i}$ $\sum_{i=1}^{i}$ $\sum_{i=1}^{i}$ $\sum_{i=1}^{i}$ $\hat{N} = \sum_{i=1}^{m} n_i \hat{C}^i \Delta^i; \quad \hat{D} = \sum_{i=1}^{m} n_i (-1)^i \hat{C}^i$ $\frac{1}{i=0}$ $\frac{1}{i=0}$ $\frac{1}{i=0}$ *n* matrices *X* and *R*
= $\sum_{i=0}^{m} n_i \hat{C}^i \Delta^i$; $\hat{D} = \sum_{i=0}^{m} n_i (-1)^i \hat{C}^i \Delta^i$ • Now, have $\begin{array}{cc} |D_{11} & D_{12} \end{array}$ $\begin{array}{cc} \Phi_1(\Delta) & G(\Delta) \end{array}$ (Δ) $D_{11} D_{12}$ $\left[\Phi_1(\Delta) \right]$ $G(\Delta)$ $\left[\begin{bmatrix} N_{11} & N_{12} \end{bmatrix}\right]$ $\begin{bmatrix} \mathbf{v}_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_1(\Delta) & \mathbf{G}(\Delta) \\ 0 & \mathbf{\Phi}_1(\Delta) \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{bmatrix}$ $\begin{array}{cc} \overline{i=0} \ D_{11} & D_{12} \end{array} \begin{bmatrix} \overline{0} & \overline{i=0} \end{bmatrix} \begin{array}{cc} \overline{0} & \overline{0} \end{array} \begin{bmatrix} \Delta \end{bmatrix} \begin{bmatrix} \overline{0} & \overline{0} \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \end{bmatrix}$ $\begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix}$ $\begin{bmatrix} \Phi_1(\Delta) & \Phi_1(\Delta) \\ 0 & \Phi_1(\Delta) \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{12} \end{bmatrix}$ $\begin{bmatrix} \overline{i=0} \ D_{11} & D_{12} \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \end{bmatrix}$ $\begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_1(\Delta) \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{bmatrix}$ $\begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_1(\Delta) \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{bmatrix}$ **Practicalities - 1**

Practicalities - 2

• Note
$$
D_{11} = N_{22}^T
$$
; $N_{11} = D_{22}^T$
\n $D_{11}\Phi_1 = N_{11} \implies N_{22}^T\Phi_1 = N_{11} \implies$ Don't need Φ_1
\n $D_{11}G + D_{12}\Phi_2 = N_{12} \implies N_{22}^T G + D_{12}\Phi_2 = N_{12}$
\n $D_{22}\Phi_2 = N_{22} \implies N_{11}^T \Phi_2 = N_{22}$
\n• So,

 $I_1^I \Phi_2 = N_{22}$ $^{1}_{22}G = N_{12} - D_{12}\Phi_2$ *T T* $N_{11}^T \Phi_2 = N$ $Ax = b$ $N_{22}^T G = N_{12} - D_1$ $\Phi_2 = N_{22}$ | $\left\{ Ax = \right\}$ $= N_{12} - D_{12} \Phi_2$ - Solve:

$$
N_{22}^{t}G = N_{12} - D_{12}\Phi_2
$$

- Form $\Phi \leftarrow \Phi_2^T(\Delta)$

$$
S \leftarrow \frac{1}{2} \Big[\Phi_2^T(\Delta)G(\Delta) + G^T(\Delta)\Phi_2(\Delta) \Big] = \frac{1}{2} \Big[(\Phi G) + (\Phi G)^T \Big]
$$

Doubling Algorithm

Doubling Algorithm:

 (2Δ) $G(2\Delta)$ (2Δ) (Δ) $G(\Delta)$ (Δ) (Δ) $G(\Delta)$ (Δ) $(2\Delta) = \Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow S(2\Delta) = \Phi_2^I(2\Delta)\Phi_1(\Delta)G(\Delta) + \Phi_2^I(2\Delta)G(\Delta)\Phi_2(\Delta)$ $(2\Delta) = \Phi_2'(\Delta)\Phi_2'(\Delta) \Rightarrow S(2\Delta) = \Phi_2'(\Delta)G(\Delta) + \Phi_2'(\Delta)S(\Delta)\Phi_2(\Delta) = S(\Delta) + \Phi(\Delta)S(\Delta)\Phi'(\Delta)$ Doubling Algorithm:
 $G(2\Delta) = G(2\Delta) = \left[\begin{matrix} \Phi_1(\Delta) & G(\Delta) \end{matrix}\right] \left[\begin{matrix} \Phi_1(\Delta) & G(\Delta) \end{matrix}\right]$ $\begin{bmatrix} G(2\Delta) \\ 2(2\Delta) \end{bmatrix} = \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2 \end{bmatrix}$ $\Phi_2(2\Delta)$ $\begin{bmatrix} = \begin{bmatrix} 0 & \Phi_2(\Delta) \end{bmatrix}$ $\begin{bmatrix} 0 & \Phi_2(\Delta) \end{bmatrix}$
 $\Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(2\Delta)\Phi_1(\Delta)G(\Delta) + \Phi_2^T(2\Delta)G(\Delta)\Phi_2$ $\Phi_2(\Delta)$ \Box $\Phi_3(\Delta)$ \Box $\Phi_2(\Delta)$ \Box $\Phi_2(\Delta)$ \Box
 $(2\Delta) = \Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(2\Delta)\Phi_1(\Delta)G(\Delta)$
 $S(2\Delta) = \Phi_2^T(\Delta)\Phi_2^T(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(\Delta)G(\Delta) + \Phi_2^T(\Delta)S(\Delta)\Phi_2$ Doublin
2 Δ) $G(2)$ Doubling Algorithm:

(2 Δ) $G(2\Delta)$ = $\left[\begin{matrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{matrix}\right]$ = $\left[\begin{matrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{matrix}\right]$ = 0 $\begin{bmatrix} 1(2\Delta) & G(2\Delta) \\ 0 & \Phi_2(2\Delta) \end{bmatrix} = \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix}$
 $(2\Delta) = \Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(2\Delta)\Phi_1(\Delta)G(\Delta) + \Phi_2^T(2\Delta)\Phi_2(\Delta)$ $\begin{equation} \left[\begin{matrix} \Sigma(\Delta) \ \Sigma_2(\Delta) \end{matrix} \right] \ \left[\begin{matrix} \Sigma \ \Sigma(\Delta) \Phi_1(\Delta) G(\Delta) + \Phi_2^T \end{matrix} \right] \end{equation}$ 0 $\Phi_2(\Delta)$ $\begin{bmatrix} 0 & \Phi_2(\Delta) \end{bmatrix}$ $\begin{bmatrix} 0 & \Phi_2(\Delta) \end{bmatrix}$ $\begin{bmatrix} 0 & \Phi_2(\Delta) \end{bmatrix}$
 $(2\Delta) = \Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(2\Delta)\Phi_1(\Delta)G(\Delta) + \Phi_2^T(2\Delta)G(\Delta)\Phi_2(\Delta)$
 $\begin{bmatrix} T_2(2\Delta) = \Phi_2^T(\Delta)\Phi_2^T(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(\$ bling Algorithm:
 $G(2\Delta)$ = $\begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \end{bmatrix}$ $\begin{bmatrix} \Phi_1(2\Delta) & G(2\Delta) \\ 0 & \Phi_2(2\Delta) \end{bmatrix} = \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix}$
 $G(2\Delta) = \Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(2\Delta)\Phi_1(\Delta)G(\Delta) + \Phi_2^T(2\Delta)G$ Doubling Algorithm:
 $\left[\begin{array}{cc} \Phi_1(2\Delta) & G(2\Delta) \\ G(2\Delta) & G(2\Delta) \end{array}\right] = \left[\begin{array}{cc} \Phi_1(\Delta) & G(\Delta) \\ G(2\Delta) & G(2\Delta) \end{array}\right]$ ■ Doubling Algorithm:
 $\begin{bmatrix} \Phi_1(2\Delta) & G(2\Delta) \\ 0 & \Phi_2(2\Delta) \end{bmatrix} = \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix}$ **Doubling Algorithm:**
 $\begin{bmatrix} \Phi_1(2\Delta) & G(2\Delta) \\ 0 & \Phi_2(2\Delta) \end{bmatrix} = \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix}$ (2 Δ) $G(2\Delta)$ = $\left[\Phi_1(\Delta)$ $G(\Delta)$ = $\Phi_2(\Delta)\right]$ = $\left[\Phi_1(\Delta)$ $G(\Delta)$ = $\Phi_2(\Delta)\right]$ = $\Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(2\Delta)\Phi_1(\Delta)G(\Delta) + \Phi_2^T(2\Delta)G(\Delta)\Phi_2(\Delta)$ $[0 \quad \Phi_2(2\Delta)] [0 \quad \Phi_2(\Delta)] [0 \quad \Phi_2(\Delta)]$
 $G(2\Delta) = \Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(2\Delta)\Phi_1(\Delta)G(\Delta) + \Phi_2^T(2\Delta)G(\Delta)\Phi_2(\Delta)$
 $\Phi_2^T(2\Delta) = \Phi_2^T(\Delta)\Phi_2^T(\Delta) \Rightarrow S(2\Delta) = \Phi_2^T(\Delta)G(\Delta) + \Phi_2^T(\Delta)S(\Delta)\Phi_2(\Delta) = S(\Delta) + \Phi(\Delta)S(\Delta)\Phi^T(\Delta)$ $G(2\Delta) = \Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2$
 $\Phi_2^T(2\Delta) = \Phi_2^T(\Delta)\Phi_2^T(\Delta) \Rightarrow S(\Phi(2\Delta) \leftarrow \Phi(\Delta)\Phi(\Delta))$

• Make use of symmetry $\Rightarrow 3M/2$ matrix multiplies to obtain *S* and Φ .

SERIES METHOD

- Substitute Taylor series for $e^{A\sigma}$, $e^{A^T\sigma}$
-

• Multiply out, group terms involving
$$
\sigma^k
$$
 and integrate:
\n
$$
S = \int_0^{\Delta} \left(I + A\sigma + \frac{A^2 \sigma^2}{2!} + \dots \right) Q \left(I + A^T \sigma + \frac{A^{T^2} \sigma^2}{2!} + \dots \right) d\sigma
$$
\n
$$
= \int_0^{\Delta} \left(Q + \left(AQ + QA^T \right) \sigma + \left(A^2 Q + 2AQA^T + QA^{T^2} \right) \frac{\sigma^2}{2} + \dots \right) d\sigma
$$
\n
$$
= Q\Delta + \left(AQ + QA^T \right) \frac{\Delta^2}{2} + \left(A^2 Q + 2AQA^T + QA^{T^2} \right) \frac{\Delta^3}{3!} + \dots
$$
\n
$$
T_1
$$
\n
$$
T_2
$$
\n
$$
T_3
$$

Series Method - 2

- <u>Note</u>: $T_k = [AT_{k-1} + T_{k-1}A^T]$ Δ/k with $T_1 = Q\Delta \Rightarrow$ terms are easy to generate.
- But, can't sum forward since adding small terms to large ones \Rightarrow round-off problems.
- Would like to sum in a reversed nested manner.
- *N* terms, *N* to be determined.
- Can we do this? Yes!!
-

• Suppose have a partial sum:
\n
$$
S = C_{N-2}Q + C_{N-1}\left[AQ + (AQ)^{T}\right] + C_{N}\left[A^{2}Q + AQA^{T} + QA^{T^{2}}\right]
$$
\n
$$
\oint_{N-2!}^{\Delta^{N-2}} \frac{\Delta^{N-1}}{N-1!} \frac{\Delta^{N}}{N!}
$$

• Multiply by *A* $(AS)^{1}$ + C_{N-3} $\begin{split} &C_{N-3}\mathcal{Q}\\ &\left(A\mathcal{Q}+\mathcal{Q}A\right)^{T}+C_{N-1}\left(A^{2}\mathcal{Q}+A\mathcal{Q}A^{T}+\left(A^{2}\mathcal{Q}\right)^{T}\right)+C_{N}\left(\ldots\right)^{T}, \end{split}$ $= C_{N-3}Q + C_{N-2}(AQ + QA)^{T} + C_{N-1}$ *T* Multiply by A
 $S \leftarrow AS + (AS)^{T} + C_{N-3}Q$ $T \left[T \right]$ $\left(\frac{\lambda^2 \Omega}{4} + \Lambda Q A^T + \left(\frac{\lambda^2 \Omega}{4} \right)^T \right]$ $AS + (AS)^{I} + C_{N-3}Q$
 $C_{N-3}Q + C_{N-2}(AQ + QA)^{T} + C_{N-1}(A^{2}Q + AQA^{T} + (A^{2}Q)^{T}) + C_{N}$ $\left(A^2Q + AQA^T + (A^2Q)^T\right) + C_N(\dots)$ $- AS + (AS)^T + C_{N-3}Q$
= $C_{N-3}Q + C_{N-2}(AQ + QA)^T + C_{N-1}(A²Q + AQA^T + (A²Q)^T) + C_N(...)$

Series Method - 3

- This pushes the series by one more term.
- So, to compute *S*:

$$
S = CNQ
$$

For $i = N-1, N-2, ..., 1$

$$
S = AS + (AS)T + CiQ
$$

End

- $C_i = \Delta^i/i!$.
- Precompute *C_i* from $C_{i+1} = C_i \Delta / (i+1); i = 1, 2, ..., N-1; C_1 = \Delta$
- Total # of matrix multiplications: *N*-1
- How to pick *N* and ∆?
	- Consider truncation error \sim ||norm of 1st neglected term||
- $||T|| \leq \frac{2||A||\Delta}{(N+1)} ||T_N||$ or $||T_{N+1}|| \leq \frac{2^N ||A||^N}{(N+1)}$ ication error ~ ||norm of 1st neglected term||
 $\frac{2||A||\Delta}{\sqrt{2}}||T_N||$ or $||T_{N+1}|| \leq \frac{2^N ||A||^N}{\sqrt{2N+1}} \Delta^N \approx \varepsilon_m$ machine accuracy on error \approx ||norm of T^{or} negli
 $\frac{\Delta}{1}$ ||T_N|| or $||T_{N+1}|| \le \frac{2^N ||A||^N}{(N+1)!}$ $N \parallel A \parallel \tilde N$ *N* $\|X_{N+1}\| \leq \frac{2\|A\|\Delta}{(N+1)} \|T_N\|$ or $\|T_{N+1}\| \leq \frac{2^N \|A\|^N}{(N+1)!} \Delta^N \approx \varepsilon_m$ ation error ~ $\|\text{norm of } 1^{\text{st}}\|$
A $\|\Delta \|T_N\|$ or $\|T_{N+1}\| \leq \frac{2^N \|A\|}{2^N}$ Consider truncation error ~ ||no
 E|| = $||T_{N+1}|| \le \frac{2||A||\Delta}{(N+1)} ||T_N||$ or $||T_L||$ Example 2 ||A||A||A||Oriented 1.4 N = 8
 $\|T_N\| \leq \frac{2\|A\|\Delta}{(N+1)} \|T_N\|$ or $\|T_{N+1}\| \leq \frac{2^N \|A\|^N}{(N+1)!} \Delta^N \approx \varepsilon$ Δ nsider truncation error ~ ||norm of 1st neglected term||
= $||T_{N+1}|| \le \frac{2||A||\Delta}{(N+1)} ||T_N||$ or $||T_{N+1}|| \le \frac{2^N ||A||^N}{(N+1)!} \Delta^N \approx \varepsilon_m$ mac tion error ~ ||norm of 1st neglected
 $\frac{A\|\Delta}{A+1\|} \|T_N\|$ or $\|T_{N+1}\| \leq \frac{2^N \|A\|^N}{(N+1)!} \Delta^N \approx$
	- So, if pick $k \ni ||A\Delta|| \leq 1/2$ need $1/(N+1)! < 10^{-6} \Rightarrow N=9$ (gives 0.27 x 10⁻⁶)
	- If pick Δ \geq $||A\Delta|| \leq 1$ need $2^{N}/(N+1)! < 10^{-6} \Rightarrow N=12$ (gives 0.6 x 10⁻⁶)
	- Also, need to compute *e ^A[∆]*. Use PADE or Chebyshev.

System Stabilization - 1

- Special Case: Idempotent matrices:
	- *A* is Idempotent \Rightarrow $A^2 = A$
	- $S(\Delta) = Q\Delta + (AQ + QA^T)(e^{\Delta} 1 \Delta) + \frac{AQA^T}{2}(e^{2\Delta} 1 + 2\Delta 4(e^{\Delta} 1))$ 1 - Δ) + $\frac{AQA^{T}}{2}$ $\left(e^{2\Delta} - 1 + 2\Delta - 4\right)$ $\left(e^{\Delta} - 1\right)$ *T T A* is Idempotent \Rightarrow $A^2 = A$
 $S(\Delta) = Q\Delta + (AQ + QA^T)(e^{\Delta} - 1 - \Delta) + \frac{AQA^T}{2}(e^{2\Delta} - 1 + 2\Delta - 4(e^{\Delta} - 1))$

Application to system stabilization

• Theorem: if $\dot{x} = Ax + Bu$ is completely controllable, then $u = -Lx(t)$ is

a stabilizing control law (i.e.,
$$
\lambda_i(A-BL) \in
$$
 LHP) with $L = B^T W^{-1}(t_f)$;
\n
$$
W(t_f) = \int_0^{t_f} e^{-A\sigma} BB^T e^{-A^T \sigma} d\sigma; \ t_f \sim \text{arbitrary (e.g., } 2|\lambda_{\text{min}}|)
$$

• Proof:

$$
\frac{d}{d\sigma} \left(e^{-A\sigma} BB^T e^{-A^T \sigma} \right) = -A \left(e^{-A\sigma} BB^T e^{-A^T \sigma} \right) - \left(e^{-A\sigma} BB^T e^{-A^T \sigma} \right) A^T
$$
\n
$$
AW + WA^T = -\int_0^{t_f} \frac{d}{d\sigma} \left(e^{-A\sigma} BB^T e^{-A^T \sigma} \right) d\sigma
$$
\n
$$
\Rightarrow \int_a^b \frac{df}{dx} dx = f(b) - f(a) = -e^{-At_f} BB^T e^{-A^T t_f} + BB^T
$$

W is PD by complete controllability, so that $2BB^T = -WW^{-1}BB^T - BB^TW^{-1}W$ $\begin{split} &BB^I~=-WW^{-1}BB^I~-BB^I\,W^{-1}W\ &\Bigl(A-BB^TW^{-1}\Bigr)W+W\Bigl(A-BB^TW^{-1}\Bigr). \end{split}$ W is PD by complete controllat
 $-2BB^T = -WW^{-1}BB^T - BB^TW^{-1}W$ *T* f $BB^T e^{-A^T t}$ *T* $\begin{aligned} B^T = -WW^{-1}BB^T-BB^TW^{-1}W \ A-BB^TW^{-1}\Big)W + W\Big(A-BB^TW^{-1}\Big)^T = -e^{-At_f}BB^Te^{-A^Tt_f} - BB^T = -Q \end{aligned}$ $\overline{A}W + W\overline{A}^T = -Q$ \Rightarrow $\left(A - BB^T W^{-1}\right)$
 $\overline{A}W + W\overline{A}^T = -Q$ $-2BB^{T} = -WW^{-1}BB^{T} - BB^{T}W^{-1}W$
 $\Rightarrow (A - BB^{T}W^{-1})W + W(A - BB^{T}W^{-1})^{T} = -e^{-At_{f}}BB^{T}e^{-A^{T}t_{f}} - BB^{T} = -Q$ $A - BB^T W^{-1} W + V$
+ $W \overline{A}^T = -Q$ since $W > 0$ and $Q \ge 0$, by Lyapunov stability theory $(\underline{x}) = \underline{x}^T W^{-1} \underline{x}$ is a Lyapunov function $(x, w > 0)$ and $(x, y) = x^T W^{-1} x$ $v(x) = x^T W$ **System Stabilization - 2**

and $\lambda_i(A) \in LHP$ if $e^{-At}Qe^{-A^Tt} \neq 0$ for all t. *T* $\bar{A}t$ $\overline{O}e^{-\overline{A}^{T}t}$ $\overline{f}_i(A) \in \text{LHP}$ if $e^{-\overline{A}t}Qe^{-\overline{A}^Tt} \neq 0$ for all t $\lambda_i(A) \in \text{LHP}$ if $e^{-\overline{A}t} Q e^{-\overline{A}^T t} \neq 0$ for thus, it is sufficient to show that $e^{-\overline{A}t}B \neq 0$ if $e^{-\overline{A}t}B=0 \implies \left[\overline{B}\overline{A}B...\overline{A}^{n-1}B\right]=0$ which contradicts the complete controllability assumption. $e^{-\overline{A}t}B=0 \Rightarrow [B\overline{A}B...\overline{A}^{n-1}B]=0$ y ontrollability assumption.
 $dv(\underline{x})/dt = -\underline{x}^T W^{-1}QW^{-1}\underline{x} \le 0$ proving that $v(\underline{x})$ is a Lyapunov function. $\Rightarrow W^{-1}A + A^TW^{-1} = -W^{-1}QW^{-1}$ $AW + WA^T = -Q$

 $(\underline{x})/dt = -\underline{x}^I W^{-1}QW^{-1}\underline{x} \leq 0$ proving that $v(\underline{x})$

i.

Alternate Forms

 \Box Alternate form for $W(t_f)$

$$
W(t_f) = e^{-At_f} \left[\int_0^{t_f} e^{A\sigma} B B^T e^{A^T \sigma} d\sigma \right] e^{-A^T t_f}
$$

\n
$$
\Rightarrow W^{-1}(t_f) = e^{A^T t_f} \left[\int_0^{t_f} e^{A\sigma} B B^T e^{A^T \sigma} d\sigma \right]^{-1} e^{At_f}
$$

Corollary: if system is not c.c, then $u = -Lx(t)$ will stabilize only the controllable modes, with $L = B^T W^{\dagger}(t_f)$ $(W^{\dagger}W) = (W^{\dagger}W); \quad (WW^{\dagger}) = (WW^{\dagger})$ † where W^{\dagger} is the pseudo inverse of W with the property: here W^{\dagger} is the pseudo inve
 $\dagger WW^{\dagger} = W^{\dagger}$; $WW^{\dagger}W = V$ † † † ; $\begin{aligned} V^{\top}WW^{\top} &= W^{\top}; \quad WW^{\top}W=W^{\top} \ W^{\top}W^{\top} &= \left(W^{\top}W\right); \quad \left(WW^{\top}\right)^{T} = \left(WW^{\top}\right)^{T} \end{aligned}$ Ilable modes, with $L = B^T W^{\dagger} (t_j W^{\dagger})$
W[†] is the pseudo inverse of *W* where W^{\dagger} is the pseudo inverse of $W^{\dagger}WW^{\dagger} = W^{\dagger}$; $WW^{\dagger}W = W$

We will discuss the computation of pseudo inverse in Lecture 7.

Bass' Method

- Def: A system is stabilizable, if there exist no uncontrollable modes.
- Corollary: if a system is c.c, use of

$$
W(t_f, \beta) = \int_0^{t_f} e^{-(A+\beta I)\sigma} B B^T e^{-(A+\beta I)^T \sigma} d\sigma
$$

W(*t_f*, β) =
$$
\int_0^{t_f} e^{-(A+βI)\sigma} BB^T e^{-(A+βI)^T \sigma} d\sigma
$$

\nwill result in closed loop poles to the left of -β
\n1) Choose $β > \frac{1}{2} \sqrt{\sum_i \sum_j (a_{ij} + a_{ji})^2}$, Gershgorin and Bendixon
\ntheorem then −(*A* + β*I*) is stable.

theorem then $-(A + \beta I)$ is stable. β .

2) $\beta > ||A||$ then $-(A + \beta I)$ is stable. em then $-(A + \beta I)$
A|| then $-(A + \beta I)$ theorem then $-(A + \beta I)$ is sta
 $\beta > ||A||$ then $-(A + \beta I)$ is sta eorem then $-(A + \beta I)$ is s
 $> ||A||$ then $-(A + \beta I)$ is s

 (A_{s},B) 2) $\beta > ||A||$ then $-(A + \beta I)$ is stable.
if (A_s, B) is controllable, then $A_sW + WA_s^T = -2BB^T$ has a PD solution $\|A\|$ then $-(A + \beta I)$ is stable.
 A_s, *B*) is controllable, then $A_sW + WA_s^T = -2BB^T$ has $= B^T W^{-1}$ if (A_s, B)
 $L = B^T W$

 σ

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$$
\begin{aligned} L &= B^T W^{-1} \\ &- \Big(A - BB^T W^{-1} \Big) W - W \Big(A - BB^T W^{-1} \Big)^T - 2 \beta W = 0 \end{aligned}
$$

 $-(A - BB^T W⁻¹)W - W(A - BB^T W⁻¹) - 2\beta W = 0$
 $(A - BB^T W⁻¹)$ will always be stable from the 2nd method of Lyapunov. $A - BB^T$
 $A - BB^T W$ $-BB^T W^-$

Summary of Lecture 3

- \Box Need for computing $\int e^{As} ds$, etc.?
- \Box How to get integrals from the exponential of a modified matrix?
- Concept of **doubling**
- Error analysis
- **Application to system stabilization**