Lecture 3:
Integrals Involving e^{At}

Prof. Krishna R. Pattipati

Dept. of Electrical and Computer Engineering
University of Connecticut
Contact: krishna@engr.uconn.edu (860) 486-2890

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Outline of Lecture 3

- What is the need for computing $\int e^{As} ds$, etc.?
- How to get integrals from the exponential of a modified matrix?
- Concept of **doubling**
- Error analysis
- Application to system stabilization

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2. E. S. Armstrong, "Series representations for the weighting matrices in sampled-data optimal linear regulator problem," IEEE Trans. on AC, Vol. AC-23, No-3, June 1978, pp. 478-479.
3. K. R. Pattipati and S. A. Shah, "On the computational aspects of the performability models of fault-tolerant computer systems," IEEE Trans. on Computers, Vol. C-39, No. 6, June-1990, pp. 832-836.

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Why Compute Integrals of e^{At} ? - 1

- What is the need for computing $\int e^{As} ds$, etc.? :

1) $\underline{\dot{x}}(t) = A\underline{x}(t) + Bu(t)$ if $\underline{u}(t)$ is piece-wise –constant over $[k\delta, (k+1)\,\delta)\ \forall k$

$$\Rightarrow\ \underline{x}(k+1) = e^{A\delta}\underline{x}(k) + \left[\int_0^{\delta} e^{A\sigma} B\, d\sigma\right]\underline{u}(k)$$

$x(k) = \underline{x}(k\delta)$, $\underline{u}(k)$ = value of $u(t)$ in the interval $[k\delta, (k+1)\delta)$.

Arises when we discretize a continuous LTI system

2) Covariance analysis of stochastic LTI systems

$$\underline{\dot{x}}(t) = A\underline{x}(t) + E\underline{w}(t); \quad \underline{X}_C(t) = E\left\{\left(\underline{x}(t) - \underline{\bar{x}}(t)\right)\left(\underline{x}(t) - \underline{\bar{x}}(t)\right)^T\right\}$$

$$\underline{X}_C(k+1) = e^{A\delta}\underline{X}_C(k)e^{A^T\delta} + \int_0^{\delta} e^{A\sigma} EWE^T e^{A^T\sigma} d\sigma$$

$$S(\delta) = \int_0^{\delta} e^{A\sigma} EWE^T e^{A^T\sigma} d\sigma \qquad \text{PD if } \left(A,\ EW^{1/2}\right) \text{ controllable}$$

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Why Compute Integrals of e^{At} ? - 2

- Suppose want to model the continuous stochastic LTI system by its discrete counterpart

$$\underline{x}(k+1) = \Phi \underline{x}(k) + \Gamma \underline{w}(k)$$

- Find Φ, Γ and cov[$\underline{w}(k)$]=W_d ∍ $\underline{X}_d(k\delta) = \underline{X}_c(k\delta)$ at the sampled points, where

$$\underline{X}_d(k+1) = \Phi \underline{X}_d(k) \Phi^T + \Gamma W_d \Gamma^T$$

- Two possibilities:

  (1) $\Phi = e^{A\delta}, \ \Gamma = I, \ W_d = S(\delta)$

  (2) $\Phi = e^{A\delta}, \ \Gamma = [S(\delta)]^{1/2}, \ W_d = I$

- We will see in Lecture 5 how to compute *square roots of positive definite matrices*

3) Integrals of the form $\int_0^{t_f} e^{A\sigma} BB^T e^{A^T \sigma} d\sigma$ are used to test the controllability of LTI systems and to solve the minimum energy control problem:

$$\min \frac{1}{2} \int_0^{t_f} \left(\underline{u}^T \underline{u}\right) dt \qquad \text{s.t. } \dot{\underline{x}} = A\underline{x} + B\underline{u} \ \text{ and } \underline{x}(t_f) = \underline{0}$$

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Why Compute Integrals of e^{At} ? - 3

Hamiltonian: $\frac{1}{2}\underline{u}^T\underline{u}+\lambda^T\left(A\underline{x}+B\underline{u}\right)$

$$\partial H/\partial\underline{u}=0 \;\Rightarrow\; \underline{u}+B^T\underline{\lambda}=0 \;\Rightarrow\; \underline{u}=-B^T\underline{\lambda}$$

$$\partial H/\partial\underline{x}=-\dot{\underline{\lambda}}=A^T\underline{\lambda};\; \underline{\lambda}\left(t_f\right)=\underline{\nu}\;\left(\underline{\nu}\text{ unknown}\right) \;\Rightarrow\; \underline{\lambda}\left(t\right)=e^{A^T\left(t_f-t\right)}\underline{\nu}$$

$$\underline{u}=-B^Te^{A^T\left(t_f-t\right)}\underline{\nu} \quad\Rightarrow x\left(t_f\right)=e^{At_f}\underline{x}_0-\int_0^{t_f}e^{At}BB^Te^{A^Tt}dt\,\underline{\nu}$$

$$\Rightarrow\underline{\nu}=\left[\int_0^{t_f}e^{At}BB^Te^{A^Tt}dt\right]^{-1}e^{At_f}\underline{x}_0$$

$$\underline{u}\left(t\right)=-B^Te^{-A^Tt}e^{A^Tt_f}\left[\int_0^{t_f}e^{At}BB^Te^{A^Tt}dt\right]^{-1}e^{At_f}\underline{x}_0$$

4) Integrals of the form $\int_0^{t_f}e^{A^T\sigma}C^TCe^{A\sigma}d\sigma$ also rise in testing the observability of LTI systems.

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Why Compute Integrals of e^{At} ? - 4

5) In sampled data regulator problem, in addition to Γ(δ) and S(δ), we also get integrals of the form:

$$M(\delta) = \int_0^{\delta} e^{A^T\sigma} Q\Gamma(\sigma)\,d\sigma \quad \text{and} \quad N(\delta) = \int_0^{\delta} \Gamma^T(\sigma) Q\Gamma(\sigma)\,d\sigma$$

6) Performability models of fault-tolerant computer systems

$$\overline{y}_{[0\ \ T]} = \underline{f}^T \left[\int_0^T e^{Q^T t}\,dt \right] \underline{p}_0$$

- So, need:
  - $\Gamma(\delta) = \int_0^{\delta} e^{A\sigma} B\,d\sigma$
  - $S(\delta) = \int_0^{\delta} e^{A\sigma} BB^T e^{A^T\sigma}\,d\sigma$ or $\int_0^{\delta} e^{A^T\sigma} C^T C e^{A\sigma}\,d\sigma$

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Computation of Γ and Φ – 1

❑ Computation of Γ and Φ: Method A

- Here δ is usually small w.r.t $1/\|A\|$, typically $0.1/\|A\|$ or $0.2/\|A\|$
- $\Psi = \int_0^\delta e^{A\sigma} d\sigma$
- So, Taylor series for $e^{A\sigma}$ is good, since $\|A\delta\| << 1$

$$\Psi = \int_0^\delta \left(I + A\sigma + \frac{A^2\sigma^2}{2!} + \ldots \right) d\sigma = \delta I + \frac{A\delta^2}{2} + \frac{A^2\delta^3}{3!} + \ldots + \frac{A^k\delta^{k+1}}{(k+1)!}$$

- Then, $\Gamma = \Psi B$, $\Phi = I + A\Psi$
- Input k to routine, $k \approx 4$, $\|A\delta\| = 0.1 \Rightarrow$ error $\approx 10^{-4}\delta/120 \approx 10^{-6}\delta$
- Function *c2d* in MATLAB computes Φ and Γ
- **Widely used method in digital control**

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Computation of Γ and Φ – 2

Computation of Γ and Φ: Method B

- $de^{At}/dt = Ae^{At};\ e^{At}|_{t=0} = I$ or $d\Phi/dt = A\Phi(t);\ \Phi(0) = I$

$$\hat{C} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix};\quad \underline{e}^{\hat{C}t} = \begin{bmatrix} \Phi_1 & \Gamma \\ 0 & \Phi_2 \end{bmatrix};\quad \frac{d}{dt}\left[e^{Ct}\right] = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Phi_1 & \Gamma \\ 0 & \Phi_2 \end{bmatrix}$$

$$\dot{\Phi}_1 = A\Phi_1 \Rightarrow \Phi_1 = e^{At};\quad \dot{\Phi}_2 = 0 \Rightarrow \Phi_2 = I$$

$$\dot{\Gamma} = A\Gamma + B\Phi_2 \Rightarrow \dot{\Gamma} = A\Gamma + B \Rightarrow \Gamma = \left[\int_0^{\delta} e^{A\sigma} d\sigma\right] B$$

- If we want $\Gamma(\delta)$, all we need to do is to find $e^{\hat{C}\delta}$ and take $\Gamma(\delta)$ as the (1,2) block of $e^{\hat{C}\delta}$.
- *Note*: δ need not be small with this approach, since we can use **shifting**, **scaling**, and **doubling** techniques to compute $e^{\hat{C}\delta}$.

$$e^{\hat{C}\delta} = \left[\hat{f}_{pade}\left(\frac{\hat{C}\delta}{2^M}\right)\right]^{2^M} \text{ or } \left[\hat{f}_{chebyshev}\left(\frac{\hat{C}\delta}{2^M}\right)\right]^{2^M};\ \hat{f}_{pade}(x) = \frac{\sum_{i=0}^{m} n_i x^i}{\sum_{i=0}^{m} (-1)^i n_i x^i},\ n_i = \frac{(2m-i)!}{2m!}\binom{m}{i}$$

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Doubling Equations for Γ and Φ

- Approach
  - Find $M \ni \left\| \hat{C}\delta / 2^M \right\| \le 1/2$
  - Let $\Delta = \delta / 2^M$
  - Find $\Phi_1(\Delta)$ and $\Gamma(\Delta)$ by PADE or Chebyshev $e^{A\Delta}$, $\left[\int_0^\Delta e^{A\sigma} d\sigma\right] B$
  - Then use the fact that: $e^{\hat{C}2t} = e^{\hat{C}t} e^{\hat{C}t}$
- ***Note:*** don't need to carry all the elements

$$\Rightarrow \begin{bmatrix} \Phi_1(2t) & \Gamma(2t) \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi_1(t) & \Gamma(t) \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_1(t) & \Gamma(t) \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \Phi_1^2(t) & \Gamma(t) + \Phi_1(t)\Gamma(t) \\ 0 & I \end{bmatrix}$$

In place computation:

$$\Gamma(2t) = \Gamma(t) + \Phi_1(t)\Gamma(t) = \left[I + \Phi_1(t)\right]\Gamma(t)$$

$$\Phi_1(2t) = \Phi_1^2(t)$$

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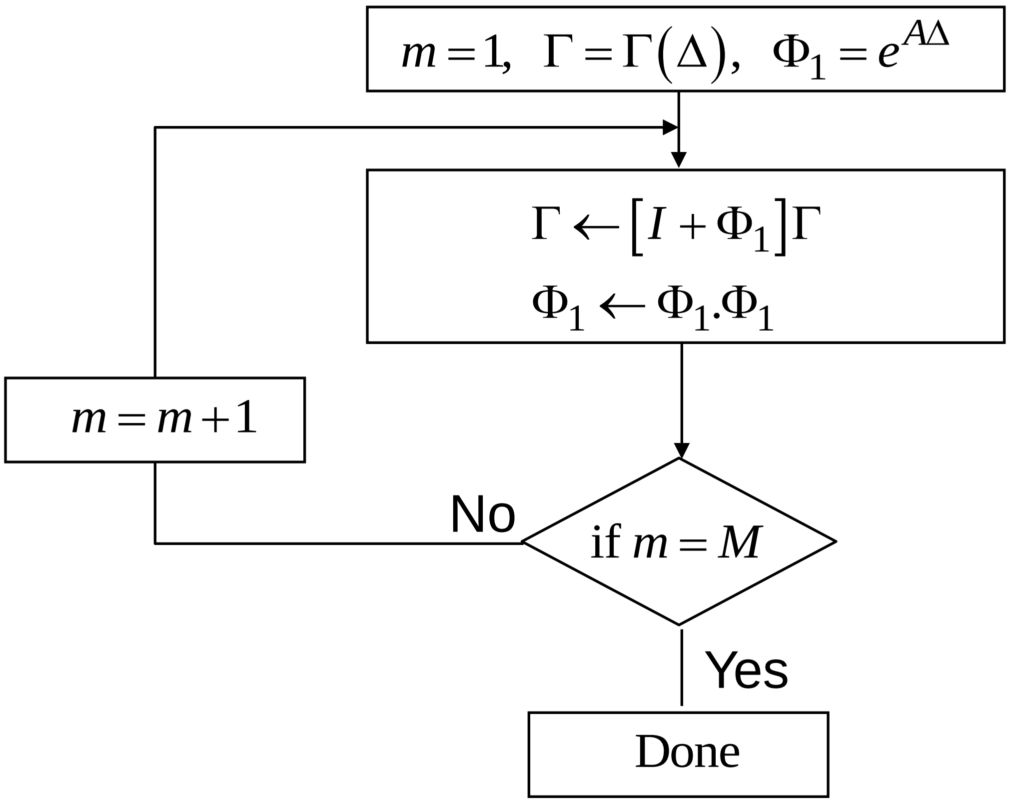
Flow Chart for Γ and Φ

- **Algorithm for summing via Doubling**



$$\Gamma = \int_0^{\delta} e^{A\sigma} B d\sigma$$

$$\Phi_1 = e^{A\delta} = \left(e^{A\Delta}\right)^{2^M}$$

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Practicalities

- Don't actually need to evaluate $\hat{C}^k$ as an $(n+m)$ by $(n+m)$ matrix in using Pade approximation. Some simplifications are possible!!

$$\hat{C} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}; \quad \hat{C}^2 = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^2 & AB \\ 0 & 0 \end{bmatrix}$$

$$\hat{C}^k = \begin{bmatrix} X_k & Y_k \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} X_k = AX_{k-1},\ X_0 = I \\ Y_k = AY_{k-1},\ Y_0 = B \end{array}$$

$\Rightarrow$ need one n x n and one n x m matrix

$$\text{PADE} \quad e^{\hat{C}\Delta} = \left[\sum_{k=0}^{m} n_k \hat{C}^k (-\Delta)^k\right]^{-1}\left[\sum_{k=0}^{m} n_k \hat{C}^k \Delta^k\right]$$

- Use Horner's rule $\Rightarrow 3m$ matrix multiples

$$\hat{D} = \begin{bmatrix} D_{11} & D_{12} \\ 0 & I \end{bmatrix}; \quad \hat{N} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & I \end{bmatrix}; \quad \begin{bmatrix} D_{11} & D_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} \Phi_1(\Delta) & \Gamma(\Delta) \\ 0 & I \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & I \end{bmatrix}$$

$\Rightarrow$ solve

$$D_{11}\Phi_1(\Delta) = N_{11} \Rightarrow D_{11}(\Phi_1 \ldots \Phi_n)_1 = (\underline{n}_1 \ldots \underline{n}_n)_{11}$$

$$D_{11}\Gamma(\Delta) + D_{12} = N_{12} \Rightarrow D_{11}(\Gamma_1 \ldots \Gamma_m) = (\underline{n}_1 - \underline{d}_1 \ldots \underline{n}_m - \underline{d}_m)$$

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Error Analysis - 1

- Same D_{11} matrix.. Can exploit this observation using the LU decomposition techniques of Lecture 4 for solving $A\underline{x}=\underline{b}$.

❑ What is the error made? (see Golub and Van Loan)

$$\left\|\Phi_1 - e^{A\delta}\right\| \le \varepsilon\delta\theta(\delta)e^{\varepsilon\delta}$$

$$\left\|\Gamma - \int_0^{\delta} e^{A\sigma} B d\sigma\right\| \le \varepsilon\delta\theta(\delta)e^{\varepsilon\delta}\left[1+\alpha\delta/2\right]$$

$$\text{where } \varepsilon = \frac{\left(2^{3-2m}\right)\left\|\hat{C}\right\|(m!)^2}{2m!(2m+1)!},\ \varepsilon \ge \left\|E_A\right\|,\ E_A = \text{error in } A$$

$$\theta(\delta) = \max_{0\le s\le\delta}\left\|e^{As}\right\|;\quad \alpha = \left\|B\right\|$$

$\Rightarrow$ can control the accuracy of the algorithm via m. e.g., choose m to satisfy

$$\frac{\left\|\Phi_1 - e^{A\delta}\right\|}{\left\|e^{A\delta}\right\|} \le \varepsilon\delta e^{\varepsilon\delta} \le \text{TOL and } \frac{\left\|\Gamma - \int_0^{\delta} e^{A\sigma} B d\sigma\right\|}{\left\|e^{A\delta}\right\|} \le \varepsilon\delta e^{\varepsilon\delta}\left[1+\alpha\delta/2\right] \le \text{TOL}$$

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Error Analysis - 2

Proof: $\left\|\Phi_1 - e^{A\delta}\right\| = \left\|e^{(A+E_A)\delta} - e^{A\delta}\right\| \le \left\|e^{A\delta}\right\|\left\|e^{E_A\delta} - I\right\| \le \theta(\delta)\varepsilon\delta e^{\varepsilon\delta}$

see Moler and Van loan for a proof that A and E_A commute

$$\Gamma - \int_0^\delta e^{A\sigma} B d\sigma = \left[\int_0^\delta e^{(A+E_A)\sigma}(B+E_B)d\sigma - \int_0^\delta e^{A\sigma} B d\sigma\right]$$

$$= \int_0^\delta e^{(A+E_A)\sigma} B d\sigma - \int_0^\delta e^{A\sigma} B d\sigma + \int_0^\delta e^{(A+E_A)\sigma} E_B d\sigma$$

$$= \int_0^\delta \left[e^{(A+E_A)\sigma} - e^{A\sigma}\right] B d\sigma + \int_0^\delta e^{(A+E_A)\sigma} E_B d\sigma$$

$$\le \int_0^\delta \left\|e^{A\sigma}\right\| \varepsilon\sigma e^{\varepsilon\sigma}\|B\| d\sigma + \int_0^\delta \left\|e^{A\sigma}\right\| \varepsilon e^{\varepsilon\delta} d\sigma$$

$$\le \int_0^\delta \theta(\delta)\varepsilon e^{\varepsilon\delta}\alpha\sigma d\sigma + \theta(\delta)\varepsilon e^{\varepsilon\delta}\delta$$

$$= \varepsilon\delta\theta(\delta)e^{\varepsilon\delta}\left[1 + \frac{\alpha\delta}{2}\right]$$

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Grammian Type Integrals - 1

- Closed form solution for idempotent matrices:
  - A is idempotent $\Rightarrow \Psi = \delta(I-A) + A\left(e^{\delta}-1\right); \quad \Phi = I + A\left(e^{\delta}-1\right)$
  - Use as test cases.
- Computation of $S = \int_0^{\delta} e^{A\sigma} Q e^{A^T\sigma} d\sigma,\ Q = Q^T$ PSD $\Rightarrow \lambda_i(Q) \geq 0$, or $\underline{x}^T Q \underline{x} \geq 0\ \forall\ \underline{x}$
  - All schemes use doubling concept, i.e., first find $\int_0^{\Delta} e^{A\sigma} Q e^{A^T\sigma} d\sigma$ where $\|A\Delta\|$ is small ($\leq 1/2$) and then get $\int_0^{\delta} e^{A\sigma} Q e^{A^T\sigma} d\sigma$ by doubling
  - Consider $\hat{C} = \begin{bmatrix} -A & Q \\ 0 & A^T \end{bmatrix};\ e^{\hat{C}t} = \begin{bmatrix} \Phi_1 & G \\ 0 & \Phi_2 \end{bmatrix}$

$$\dot{\Phi}_1 = -A\Phi_1 \quad \Rightarrow \quad \Phi_1 = e^{-At};$$

$$\dot{G} = -AG + Q\Phi_2;\quad G(0) = 0$$

$$\dot{\Phi}_2 = A^T\Phi_2 \Rightarrow \Phi_2(t) = e^{A^T t} = \left[\Phi_1^{-1}(t)\right]^T$$

$$\Rightarrow \quad G(t) = \int_0^t e^{-A(t-\sigma)} Q e^{A^T\sigma} d\sigma = e^{-At}\int_0^t e^{A\sigma} Q e^{A^T\sigma} d\sigma$$

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Error Analysis

$S(\Delta) = \Phi_2^T(\Delta) G(\Delta)$ MUST MAKE THIS SYMMETRIC

$$S(\Delta) = \frac{1}{2}\left[\Phi_2^T(\Delta) G(\Delta) + G^T(\Delta)\Phi_2(\Delta)\right]$$

$\Rightarrow$ Compute $\Phi_1(\Delta)$ and $S(\Delta)$ from $\exp\{\hat{C}\Delta\}$

$\Rightarrow$ PADE or Chebyshev approximation

- What is the error made?

$$\left\|\Phi(\Delta) - e^{A\delta}\right\| \le \varepsilon\Delta\theta(\Delta)e^{\varepsilon\delta}$$ same as earlier

$$\left\|S(\Delta) - \int_0^{\Delta} e^{A\sigma} Q e^{A^T\sigma} d\sigma\right\| \le \varepsilon\Delta\theta^2(\Delta)e^{2\varepsilon\Delta}\left(1+\alpha_q\Delta\right), \quad \alpha_q = \|Q\|$$

Proof: $S(\Delta) - \int_0^{\Delta} e^{A\sigma} Q e^{A^T\sigma} d\sigma = \int_0^{\Delta} e^{(A+E_A)\sigma}\left(Q+E_Q\right)e^{(A+E_A)^T\sigma} d\sigma - \int_0^{\Delta} e^{A\sigma} Q e^{A^T\sigma} d\sigma$

$$= \int_0^{\Delta}\left[e^{(A+E_A)\sigma} - e^{A\sigma}\right] Q \left[e^{(A+E_A)^T\sigma} - e^{A^T\sigma}\right] d\sigma$$

$$+ \int_0^{\Delta} e^{A\sigma} Q \left[e^{(A+E_A)^T\sigma} - e^{A^T\sigma}\right] d\sigma + \int_0^{\Delta} e^{(A+E_A)\sigma} E_Q e^{(A+E_A)^T\sigma} d\sigma$$

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Practicalities - 1

Take norms $\leq \int_0^{\Delta} \varepsilon\sigma e^{2\varepsilon\Delta}\alpha_q\theta^2(\Delta)d\sigma + \int_0^{\Delta} \varepsilon\sigma\theta^2(\Delta)e^{\varepsilon\Delta}d\sigma + \int_0^{\Delta} \theta^2(\Delta)\varepsilon e^{2\varepsilon\Delta}d\sigma$

$$= \theta^2(\Delta)\varepsilon e^{2\varepsilon\Delta}\left[\Delta + \frac{\alpha_q\Delta^2}{2} + e^{-\varepsilon\Delta}\frac{\alpha_q\Delta^2}{2}\right]$$

$$\leq \theta^2(\Delta)\varepsilon e^{2\varepsilon\Delta}\left[\Delta + \alpha_q\Delta^2\right]$$

- Computation of $e^{\hat{C}\Delta}$ can be simplified

$$\hat{C}^k = \begin{bmatrix} (-1)^k X_k & R_k \\ 0 & X_k^T \end{bmatrix} \Rightarrow \begin{array}{c} X_k = AX_{k-1} \\ R_k = -AR_{k-1} + QX_{k-1}^T \end{array}$$

- Need 2 n x n matrices X and R
- Compute $\hat{N} = \sum_{i=0}^{m} n_i\hat{C}^i\Delta^i; \quad \hat{D} = \sum_{i=0}^{m} n_i(-1)^i\hat{C}^i\Delta^i$
- Now, have $\begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}\begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_1(\Delta) \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{bmatrix}$

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Practicalities - 2

- Note $$D_{11} = N_{22}^T; \quad N_{11} = D_{22}^T$$
$$D_{11}\Phi_1 = N_{11} \;\Rightarrow\; N_{22}^T\Phi_1 = N_{11} \;\Rightarrow\; \text{Don't need } \Phi_1$$
$$D_{11}G + D_{12}\Phi_2 = N_{12} \;\Rightarrow\; N_{22}^T G + D_{12}\Phi_2 = N_{12}$$
$$D_{22}\Phi_2 = N_{22} \;\Rightarrow\; N_{11}^T\Phi_2 = N_{22}$$
- So,
  - Solve: $$\left.\begin{array}{l} N_{11}^T\Phi_2 = N_{22} \\ N_{22}^T G = N_{12} - D_{12}\Phi_2 \end{array}\right\} A\underline{x} = \underline{b}$$
  - Form $$\Phi \leftarrow \Phi_2^T(\Delta)$$
$$S \leftarrow \frac{1}{2}\left[\Phi_2^T(\Delta)G(\Delta) + G^T(\Delta)\Phi_2(\Delta)\right] = \frac{1}{2}\left[(\Phi G) + (\Phi G)^T\right]$$

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Doubling Algorithm

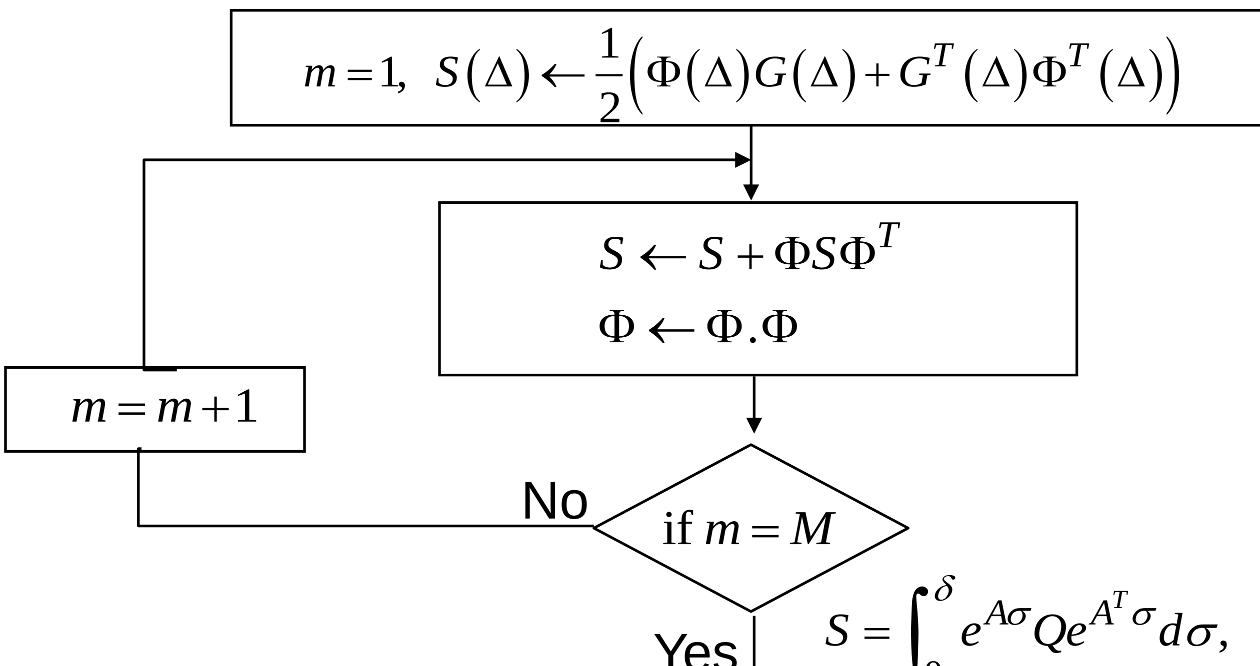
- Doubling Algorithm:

$$\begin{bmatrix} \Phi_1(2\Delta) & G(2\Delta) \\ 0 & \Phi_2(2\Delta) \end{bmatrix} = \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix} \begin{bmatrix} \Phi_1(\Delta) & G(\Delta) \\ 0 & \Phi_2(\Delta) \end{bmatrix}$$

$$G(2\Delta) = \Phi_1(\Delta)G(\Delta) + G(\Delta)\Phi_2(\Delta) \Rightarrow \quad S(2\Delta) = \Phi_2^T(2\Delta)\Phi_1(\Delta)G(\Delta) + \Phi_2^T(2\Delta)G(\Delta)\Phi_2(\Delta)$$

$$\Phi_2^T(2\Delta) = \Phi_2^T(\Delta)\Phi_2^T(\Delta) \Rightarrow \quad S(2\Delta) = \Phi_2^T(\Delta)G(\Delta) + \Phi_2^T(\Delta)S(\Delta)\Phi_2(\Delta) = S(\Delta) + \Phi(\Delta)S(\Delta)\Phi^T(\Delta)$$

$$\Phi(2\Delta) \leftarrow \Phi(\Delta)\Phi(\Delta)$$



$$S = \int_0^\delta e^{A\sigma} Q e^{A^T\sigma} d\sigma, \quad \Phi = e^{A\delta} = \left(e^{A\Delta}\right)^{2^M}$$

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Series Method - 1

- Make use of symmetry $\Rightarrow$ $3M/2$ matrix multiplies to obtain S and Φ.

❑ SERIES METHOD

- Substitute Taylor series for $e^{A\sigma}$, $e^{A^T\sigma}$
- Multiply out, group terms involving σ^k and integrate:

$$S = \int_0^{\Delta}\left(I + A\sigma + \frac{A^2\sigma^2}{2!} + \dots\right)Q\left(I + A^T\sigma + \frac{A^{T^2}\sigma^2}{2!} + \dots\right)d\sigma$$

$$= \int_0^{\Delta}\left(Q + \left(AQ + QA^T\right)\sigma + \left(A^2Q + 2AQA^T + QA^{T^2}\right)\frac{\sigma^2}{2} + \dots\right)d\sigma$$

$$= \underbrace{Q\overset{C_1}{\Delta}}_{T_1} + \underbrace{\left(AQ + QA^T\right)\overset{C_2}{\frac{\Delta^2}{2}}}_{T_2} + \underbrace{\left(A^2Q + 2AQA^T + QA^{T^2}\right)\overset{C_3}{\frac{\Delta^3}{3!}}}_{T_3} + \dots$$

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Series Method - 2

- Note: $T_k=[AT_{k-1}+T_{k-1}A^T]\Delta/k$ with $T_1=Q\Delta \Rightarrow$ terms are easy to generate.
- But, can't sum forward since adding small terms to large ones $\Rightarrow$ round-off problems.
- Would like to sum in a reversed nested manner.
- N terms, N to be determined.
- Can we do this? Yes!!
- Suppose have a partial sum:

$$S = C_{N-2}Q + C_{N-1}\left[AQ + (AQ)^T\right] + C_N\left[A^2Q + AQA^T + QA^{T^2}\right]$$

$$C_{N-2} \updownarrow \frac{\Delta^{N-2}}{N-2!}, \quad C_{N-1} \updownarrow \frac{\Delta^{N-1}}{N-1!}, \quad C_N \updownarrow \frac{\Delta^{N}}{N!}$$

- Multiply by A

$$S \leftarrow AS + (AS)^T + C_{N-3}Q$$

$$= C_{N-3}Q + C_{N-2}(AQ + QA)^T + C_{N-1}\left(A^2Q + AQA^T + (A^2Q)^T\right) + C_N(\ldots)$$

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Series Method - 3

- This pushes the series by one more term.
- So, to compute S:

  $S = C_N Q$

  For $i = N\text{-}1, N\text{-}2, \ldots, 1$

  $S = AS + (AS)^T + C_i Q$

  End
- $C_i = \Delta^i / i!$.
- Precompute C_i from $C_{i+1} = C_i \Delta / (i+1)$; $i = 1, 2, \ldots, N\text{-}1$; $C_1 = \Delta$
- Total # of matrix multiplications: N-1

❑ How to pick N and Δ?

- Consider truncation error ~ ||norm of 1st neglected term||

$$\|E\| = \|T_{N+1}\| \le \frac{2\|A\|\Delta}{(N+1)}\|T_N\| \quad \text{or} \quad \|T_{N+1}\| \le \frac{2^N \|A\|^N}{(N+1)!}\Delta^N \approx \varepsilon_m \quad \text{machine accuracy}$$

- So, if pick $k \ni \|A\Delta\| \le 1/2$ need $1/(N+1)! < 10^{-6} \Rightarrow N=9$ (gives 0.27×10^{-6})
- If pick $\Delta \ni \|A\Delta\| \le 1$ need $2^N/(N+1)! < 10^{-6} \Rightarrow N=12$ (gives 0.6×10^{-6})
- Also, need to compute $e^{A\Delta}$. Use PADE or Chebyshev.

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System Stabilization - 1

- Special Case: Idempotent matrices:
  - A is Idempotent $\Rightarrow A^2=A$
  - $S(\Delta)=Q\Delta+\left(AQ+QA^T\right)\left(e^{\Delta}-1-\Delta\right)+\dfrac{AQA^T}{2}\left(e^{2\Delta}-1+2\Delta-4\left(e^{\Delta}-1\right)\right)$
- Application to system stabilization
  - Theorem: if $\dot{\underline{x}}=A\underline{x}+B\underline{u}$ is completely controllable, then $\underline{u}=-L\underline{x}(t)$ is a stabilizing control law (i.e., $\lambda_i(A\text{-}BL) \in$ LHP) with $L = B^TW^{-1}(t_f)$;

$$W\left(t_f\right)=\int_0^{t_f} e^{-A\sigma}BB^Te^{-A^T\sigma}d\sigma;\ \ t_f \sim \text{arbitrary}\left(\text{e.g., } 2/\left|\lambda_{\min}\right|\right)$$

  - Proof:

$$\frac{d}{d\sigma}\left(e^{-A\sigma}BB^Te^{-A^T\sigma}\right)=-A\left(e^{-A\sigma}BB^Te^{-A^T\sigma}\right)-\left(e^{-A\sigma}BB^Te^{-A^T\sigma}\right)A^T$$

$$AW+WA^T=-\int_0^{t_f}\frac{d}{d\sigma}\left(e^{-A\sigma}BB^Te^{-A^T\sigma}\right)d\sigma$$

$$\Rightarrow \int_a^b \frac{df}{dx}dx=f(b)-f(a)=-e^{-At_f}BB^Te^{-A^Tt_f}+BB^T$$

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System Stabilization - 2

W is PD by complete controllability, so that

$$-2BB^T = -WW^{-1}BB^T - BB^TW^{-1}W$$

$$\Rightarrow \left(A - BB^TW^{-1}\right)W + W\left(A - BB^TW^{-1}\right)^T = -e^{-At_f}BB^Te^{-A^Tt_f} - BB^T = -Q$$

$$\bar{A}W + W\bar{A}^T = -Q$$

since $W > 0$ and $Q \geq 0$, by Lyapunov stability theory

$v\left(\underline{x}\right) = \underline{x}^TW^{-1}\underline{x}$ is a Lyapunov function

and $\lambda_i\left(A\right) \in$ LHP if $e^{-\bar{A}t}Qe^{-\bar{A}^Tt} \neq 0$ for all t.

thus, it is sufficient to show that $e^{-\bar{A}t}B \neq 0$

$$AW + WA^T = -Q$$
$$\Rightarrow W^{-1}A + A^TW^{-1} = -W^{-1}QW^{-1}$$

if $e^{-\bar{A}t}B = 0 \;\Rightarrow\; \left[B\,\bar{A}B \ldots \bar{A}^{n-1}B\right] = 0$ which contradicts the complete controllability assumption.

$dv\left(\underline{x}\right)/dt = -\underline{x}^TW^{-1}QW^{-1}\underline{x} \leq 0$ proving that $v\left(\underline{x}\right)$ is a Lyapunov function.

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Alternate Forms

- Alternate form for $W(t_f)$

$$W(t_f) = e^{-At_f}\left[\int_0^{t_f} e^{A\sigma}BB^T e^{A^T\sigma}d\sigma\right]e^{-A^T t_f}$$

$$\Rightarrow W^{-1}(t_f) = e^{A^T t_f}\left[\int_0^{t_f} e^{A\sigma}BB^T e^{A^T\sigma}d\sigma\right]^{-1} e^{At_f}$$

- Corollary: if system is not c.c, then $\underline{u} = -L\underline{x}(t)$ will stabilize only the controllable modes, with $L = B^T W^{\dagger}(t_f)$

where $W^{\dagger}$ is the pseudo inverse of W with the property:

$$W^{\dagger}WW^{\dagger} = W^{\dagger}; \quad WW^{\dagger}W = W$$

$$\left(W^{\dagger}W\right)^T = \left(W^{\dagger}W\right); \quad \left(WW^{\dagger}\right)^T = \left(WW^{\dagger}\right)$$

We will discuss the computation of pseudo inverse in Lecture 7.

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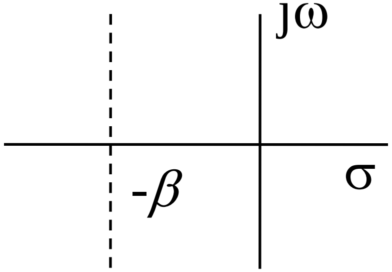
Bass' Method

- <u>Def</u>: A system is stabilizable, if there exist no uncontrollable modes.
- Corollary: if a system is c.c, use of

$$W\left(t_f,\beta\right)=\int_0^{t_f} e^{-(A+\beta I)\sigma}BB^T e^{-(A+\beta I)^T\sigma}d\sigma$$

will result in closed loop poles to the left of $-\beta$



1) Choose $\beta>\frac{1}{2}\sqrt{\sum_i\sum_j\left(a_{ij}+a_{ji}\right)^2}$, Gershgorin and Bendixon theorem then $-\left(A+\beta I\right)$ is stable.

2) $\beta>\|A\|$ then $-\left(A+\beta I\right)$ is stable.

if $\left(A_s,B\right)$ is controllable, then $A_sW+WA_s^T=-2BB^T$ has a PD solution

$$L=B^TW^{-1}$$

$$-\left(A-BB^TW^{-1}\right)W-W\left(A-BB^TW^{-1}\right)^T-2\beta W=0$$

$\left(A-BB^TW^{-1}\right)$ will always be stable from the 2nd method of Lyapunov.

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Summary of Lecture 3

- Need for computing $\int e^{As} ds$, etc.?
- How to get integrals from the exponential of a modified matrix?
- Concept of **doubling**
- Error analysis
- Application to system stabilization

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