Lecture 8: Recursive Square-Root & QR Updating for Least Squares and Kalman Filtering

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ECE 6435

Adv Numerical Methods in Sci Comp

Fall 2008

October 15, 2008

Lecture Outline

- Recursive (sequential) Least Squares
- Sequential LDL^T Factorization updates
- Sequential QR updates
- Application to Kalman filtering

Recursive Least Squares (RLS)

- Suppose we have measurements $b_i = \underline{a}_i^T \underline{x},\ b_i$ scalar for $i = 1, 2, \ldots, k,\ \underline{x} \in R^n$ unknowns (occurs in many applications, e.g., fitting an n^{th} order Polynomial

$$\begin{bmatrix} \leftarrow \underline{a}_1^T \rightarrow \\ \leftarrow \underline{a}_2^T \rightarrow \\ . \\ . \\ \leftarrow \underline{a}_k^T \rightarrow \end{bmatrix} \underline{x} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_k \end{bmatrix}$$ assume $k > n$ and A has full column rank n

- Objective is to find $\underline{x}_{\mathrm{LS}}$ to minimize the mean-squared error (MMSE)

MSE: $\left\| A_k \underline{x} - \underline{b}_k \right\|_2^2$

MMSE: Find $\underline{x}_{\mathrm{LS}} \ni \left\| A_k \underline{x} - \underline{b}_k \right\|_2^2$ is a minimum

Know $\underline{\hat{x}}_{\mathrm{LS}} = A_k^{\dagger} \underline{b}_k = \left(A_k^T A_k \right)^{-1} A_k^T \underline{b}_k \triangleq \underline{\hat{x}}_k$

Setting the Stage

- Suppose now we make a (k+1)st measurement $b_{k+1} = \underline{a}_{k+1}^T \underline{x}$

- **Q:** Can we update our previous estimates in light of b_{k+1} without recomputing $A_{k+1}^T A_{k+1}$ or Householder or Givens or Gram-Schmidt?

- **A:** Yes! This is precisely what is done in RLS, Kalman filtering, etc.

- How does recursive least-squares (RLS) work ?

  - Let $\hat{\underline{x}}_k$ be an estimate of $\underline{x}$ using $b_1, b_2, \ldots, b_k = \left(A_k^T A_k\right)^{-1} A_k^T \underline{b}_k$
  - Let $\hat{\underline{x}}_{k+1}$ be an estimate of $\underline{x}$ using $b_1, b_2, \ldots, b_{k+1}$

    $$\hat{\underline{x}}_{k+1} = \left(A_{k+1}^T A_{k+1}\right)^{-1} A_{k+1}^T \underline{b}_{k+1}$$

  - Define $P_{k+1} = \left(A_{k+1}^T A_{k+1}\right)^{-1}$ and $P_k = \left(A_k^T A_k\right)^{-1}$
  - In RLS, we estimate $\hat{\underline{x}}_{k+1}$ from $\hat{\underline{x}}_k$ and b_{k+1}, and P_{k+1} from P_k and $\underline{a}_{k+1}$

Sequential Update of Covariance Matrix

- Mechanics of RLS process
  - Consider $A_{k+1}A_{k+1}^T$:

$$A_{k+1}^T A_{k+1} = A_k^T A_k + \underline{a}_{k+1}\underline{a}_{k+1}^T$$

$$\Rightarrow P_{k+1}^{-1} = P_k^{-1} + \underline{a}_{k+1}\underline{a}_{k+1}^T \quad ; P_k^{-1} \sim \text{is the so called information matrix}$$

so, every succesive measurement adds "INFORMATION"

  - Key: Sherman-Morrison-Woodbury Formula

- Consider three matrices: A is *n* x *n*, B is *n* x *m* C = *n* x *m*
  - Then Sherman-Morrison-Woodbury Formula gives:

$$\left(A + BC^T\right)^{-1} = A^{-1} - A^{-1}B\left(I + C^T A^{-1} B\right)^{-1} C^T A^{-1}$$

$$\Rightarrow P_{k+1} = P_k - P_k \underline{a}_{k+1}\left(\underline{a}_{k+1}^T P_k \underline{a}_{k+1} + 1\right)^{-1} \underline{a}_{k+1}^T P_k$$

$\Rightarrow$ Requires scalar inversion

$$\Rightarrow P_{k+1} = P_k - P_k \underline{a}_{k+1}\underline{a}_{k+1}^T P_k / \left(1 + \underline{a}_{k+1}^T P_k \underline{a}_{k+1}\right)$$

Sequential Update of Estimate

- To compute $\hat{\underline{x}}_{k+1}$

$$\hat{\underline{x}}_{k+1} = P_{k+1}\begin{bmatrix} A_k^T & \underline{a}_{k+1} \end{bmatrix}\begin{bmatrix} \underline{b}_k \\ b_{k+1} \end{bmatrix} = P_{k+1}\left[A_k^T \underline{b}_k + b_{k+1}\underline{a}_{k+1} \right]$$

$$\hat{\underline{x}}_{k+1} = \left[P_k - \frac{P_k \underline{a}_{k+1}\underline{a}_{k+1}^T P_k}{1+\underline{a}_{k+1}^T P_k \underline{a}_{k+1}} \right]\left[A_k^T \underline{b}_k + b_{k+1}\underline{a}_{k+1} \right]$$

$$= \hat{\underline{x}}_k + P_k \underline{a}_{k+1} b_{k+1} - \frac{P_k \underline{a}_{k+1}\underline{a}_{k+1}^T P_k \underline{a}_{k+1}}{1+\underline{a}_{k+1}^T P_k \underline{a}_{k+1}} b_{k+1} - \frac{P_k \underline{a}_{k+1}\underline{a}_{k+1}^T \overbrace{P_k A_k^T \underline{b}_k}^{\hat{\underline{x}}_k}}{1+\underline{a}_{k+1}^T P_k \underline{a}_{k+1}}$$

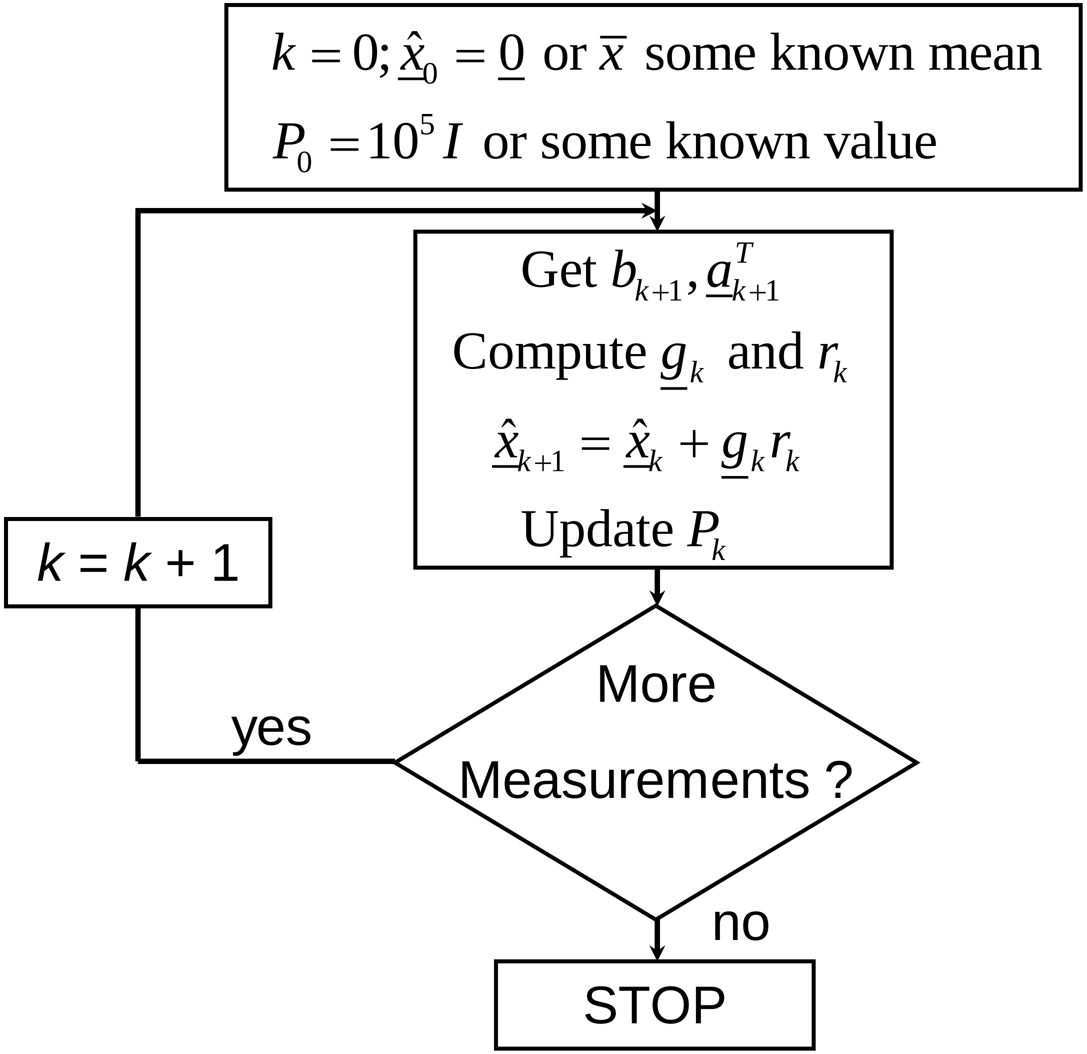
$$= \hat{\underline{x}}_k + \underbrace{\frac{P_k \underline{a}_{k+1}}{1+\underline{a}_{k+1}^T P_k \underline{a}_{k+1}}}_{\text{Gain vector},\ \underline{g}_k}\underbrace{\left[b_{k+1} - \underline{a}_{k+1}^T \hat{\underline{x}}_k \right]}_{\text{Residual or innovation},\ r_k}$$

$$= \hat{\underline{x}}_k + \underline{g}_k r_k = \left[I - \underline{g}_k \underline{a}_{k+1}^T \right]\hat{\underline{x}}_k + \underline{g}_k b_{k+1}$$

$\Rightarrow$ $\hat{\underline{x}}_{k+1}$ is a weighted sum of previous estimate and current measurement

$\Rightarrow$ This is similar to the measurement update of a Kalman filter

RLS is Simple to Implement



Round-off Error Issues

- Major problem: Negative sign in P_k equation causes P_k to go indefinite due to round-off errors (e.g., negative diagonals)

- Other formulae to overcome indefiniteness
  1. Joseph's form:

$$P_{k+1} = \left(I - \underline{g}_k \underline{a}_{k+1}^T\right) P_{k+1} \left(I - \underline{g}_k \underline{a}_{k+1}^T\right)^T + \underline{g}_k \underline{g}_k^T$$

     This transformation requires twice the number of operations over the ordinary RLS

  2. Square-root or LDL^T update
     – Idea: force P_k and P_{k+1} to be PD

$$\text{i.e., write } \quad P_k = L_k D_k L_k^T \text{ via "LDL}^T\text{" factorization}$$

$$L_k = \text{ unit lower } \Delta;$$

$$D_k = diag\left(d_i\right), d_i > 0$$

Setting the Stage for LDL^T Update

- *Q*: Can we go from $\begin{bmatrix} L_k \\ D_k \end{bmatrix} \rightarrow \begin{bmatrix} L_{k+1} \\ D_{k+1} \end{bmatrix}$ recursively ?
- *A*: Yes, but slightly complicated
- Simplicity of notation, let

$$P_k = LDL^T, \quad P_{k+1} = \bar{L}\bar{D}\bar{L}^T, \quad \underline{a}_{k+1} = \underline{a}, \text{ then}$$

$$P_{k+1} = LDL^T - \left(LDL^T \underline{a}\,\underline{a}^T LDL^T\right) / \left(1 + \underline{a}^T LDL^T \underline{a}\right)$$

- To simplify the expression for P_{k+1}, let $\underline{f} = L^T \underline{a}$, then

$$P_{k+1} = L\left[D - \frac{\underline{v}\,\underline{v}^T}{1 + \underline{f}^T D \underline{f}}\right] L^T; \underline{v} = D\underline{f} \text{ or } v_i = d_i f_i$$

- So, if we can find $\tilde{L}\tilde{D}\tilde{L}^T$ of the terms in brackets, then we have solved the problem:

$$\bar{L} = L\tilde{L} \text{ and } \bar{D} = \tilde{D}$$

LDL^T Update with Rank 1 Correction

- This is a special case of the following more general problem:

  "Given $A = LDL^T$, find $\bar{L}\bar{D}\bar{L}^T$ factorization of $A + \sigma \underline{v}\underline{v}^T$"

  $\Rightarrow$ This is basically a problem of updating LDL^T factorizations of a rank-one corrected matrix

- Problem: updating LDL^T factorizations of a rank-one corrected matrix
  – Starting with

$$A = \sum_{i=1}^{n} d_i \underline{l}_i \underline{l}_i^T \text{ with } \underline{l}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ l_{i+1,i} \\ l_{n,i} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow l_{ii} \\ \\ \\ \end{matrix}$$

we want to obtain factorization of $A + \sigma \underline{v}\underline{v}^T = \sum_{i=1}^{n} \bar{d}_i \bar{\underline{l}}_i \bar{\underline{l}}_i^T$

$$\text{i.e.,} \begin{pmatrix} \underline{l}_i \\ d_i \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\underline{l}}_i \\ \bar{d}_i \end{pmatrix}$$

Four Cases for LDL^T Updates

- Consider 4 cases: From general → specific to RLS

1) $\sigma > 0$ arbitrary

2) $\sigma < 0; \sigma = -1/\alpha;\ \alpha > 0$ arbitrary

3) $A = Diag \Rightarrow L = I$

4) $\sigma = -1/\alpha;\ \alpha = 1 + \underline{f}^T D \underline{f};\ \underline{f} = L^T \underline{a}$; special α

Case 1: $\sigma > 0$ arbitrary

- Develop algorithm one column at a time
- Let $\sigma_1 = \sigma; \underline{v}_1 = \underline{v}$. Then

$$\sum_{i=1}^{n} \underline{l}_i \underline{l}_i^T d_i + \sigma_1 \underline{v}_1 \underline{v}_1^T = \underline{l}_1 \underline{l}_1^T d_1 + \sigma_1 \underline{v}_1 \underline{v}_1^T + \sum_{i=2}^{n} d_i \underline{l}_i \underline{l}_i^T$$

- *Q*: can we write

$$\underline{l}_1 \underline{l}_1^T d_1 + \sigma_1 \underline{v}_1 \underline{v}_1^T = \bar{\underline{l}}_1 \bar{\underline{l}}_1^T \bar{d}_1 + \sigma_2 \underline{v}_2 \underline{v}_2^T \qquad (*)$$

where

$$\underline{l}_1 = \begin{bmatrix} 1 \\ * \\ * \\ \cdot \\ \cdot \\ * \end{bmatrix} \quad \text{and} \quad \underline{v}_2 = \begin{bmatrix} 0 \\ \times \\ \times \\ \cdot \\ \cdot \\ \times \end{bmatrix} \Rightarrow \text{first component of } \underline{v}_2, v_{21} = 0.$$

Case 1 (contd.)

- If so, we can generate a recursive scheme to update $\begin{pmatrix} \underline{l}_i \\ d_i \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\underline{l}}_i \\ \bar{d}_i \end{pmatrix}$
- Matrix on LHS has at most rank 2 and range space spanned by $\underline{l}_1$ and $\underline{v}_1$
  $\Rightarrow \underline{l}_1$ and $\underline{v}_2$ must be linear combinations of $\underline{l}_1$ and $\underline{v}_1$
- Let

$$\underline{v}_2 = \underline{v}_1 - v_{11}\underline{l}_1 \text{ since } \underline{v}_{21} = 0 \text{ and } l_{11} = 1$$

$$\bar{\underline{l}}_1 = \underline{l}_1 + \beta_1 \underline{v}_2 = (1-\beta_1 v_{11})\underline{l}_1 + \beta_1 \underline{v}_1$$

where β_1 is arbitrary and is to be determined

- Must solve for "unknowns" $\bar{d}_1, \sigma_2$ and $\beta_1 \ni$ Eqn.$(*)$ is satisfied
- Need 3 eqns. Since we have 3 unknowns: $\bar{d}_1, \sigma_2$ and β_1
- Substitute for v_2 and $\bar{\underline{l}}_1$ into * and equate coefficients.

$$d_1\underline{l}_1\underline{l}_1^T + \sigma_1\underline{v}_1\underline{v}_1^T = \left[(1-\beta_1 v_{11})\underline{l}_1 + \beta_1\underline{v}_1\right]\bar{d}_1\left[(1-\beta_1 v_{11})\underline{l}_1 + \beta_1\underline{v}_1\right]^T$$
$$+\sigma_2\left(\underline{v}_1 - v_{11}\underline{l}_1\right)\left(\underline{v}_1 - v_{11}\underline{l}_1\right)^T$$

Case 1 (contd.)

coeff. of $\underline{v}_1\underline{v}_1^T : \sigma_1 = \bar{d}_1\beta_1^2 + \sigma_2 \Rightarrow \sigma_2 = \sigma_1 - \bar{d}_1\beta_1^2$

coeff. of $\underline{l}_1\underline{v}_1^T : 0 = 2\bar{d}_1\left(1-\beta_1 v_{11}\right)\beta_1 - 2\sigma_2 v_{11} = 0 \Rightarrow \beta_1 = \sigma_1 v_{11} / \bar{d}_1$

coeff. of $\underline{l}_1\underline{l}_1^T : d_1 = \left(1-\beta_1 v_{11}\right)^2 \bar{d}_1 + \sigma_2 v_{11}^2$

$$\begin{aligned} &= \left(1-\beta_1 v_{11}\right)^2 \bar{d}_1 + \sigma_1 v_{11}^2 - \bar{d}_1 v_{11}^2 \beta_1^2 \\ &= \bar{d}_1 - 2\beta_1 v_{11}\bar{d}_1 + \sigma_1 v_{11}^2 \\ &= \bar{d}_1 - 2\sigma_1 v_{11}^2 + \sigma_1 v_{11}^2 \end{aligned}$$

$\Rightarrow \; \bar{d}_1 = d_1 + \sigma_1 v_{11}^2$. Also, note $\sigma_2 = \sigma_1 - \bar{d}_1\beta_1^2 = \sigma_1\left(1 - \frac{\sigma_1 v_{11}^2}{\bar{d}_1}\right) = \sigma_1 \frac{d_1}{\bar{d}_1}$

$\Rightarrow$ So, we compute $\bar{d}_1, \sigma_2$ and β_1 in that order

- Next, repeat with $\underline{l}_2\underline{l}_2^T d_2 + \sigma_2\underline{v}_2\underline{v}_2^T \rightarrow \bar{\underline{l}}_2\bar{\underline{l}}_2^T d_2 + \sigma_3\underline{v}_3\underline{v}_3^T$

  where $v_{31} = v_{32} = 0$

Update Algorithm for Case 1

Initialize $\sigma_1 = \sigma; \underline{v}_1 = \underline{v}$

For $k = 1, 2, \ldots, n-1$ DO

1) $\bar{d}_k = d_k + \sigma_k v_{kk}^2$

2) $\beta_k = \sigma_k v_{kk} / \bar{d}_k$

3) $\sigma_{k+1} = \sigma_k \dfrac{d_k}{\bar{d}_k}$

4) $\underline{v}_{k+1} = \underline{v}_k - v_{kk} \underline{l}_k$ note: $\underline{v}_{k+1} = \begin{bmatrix} 0 \\ . \\ 0 \\ \times \\ \times \end{bmatrix} \leftarrow k.$

So, we need to compute elements $(k+1, \ldots n)$ only

5) $\underline{\bar{l}}_k = \underline{l}_k + \beta_k \underline{v}_{k+1}$

End DO

$\bar{d}_n = d_n + \sigma_n v_{nn}^2$

Case 2: $\sigma < 0$ (as in RLS)

- In this case, we may end up in a situation where $\bar{d}_k < 0$ in step 1.
- This may be due to round-off or near rank degeneracy of P_k
- Need slightly different formulae:

Let $\sigma_k = -1/\alpha_k$; assume $\alpha_k > 0$ with $\alpha_1 = \alpha = -1/\sigma$

- Step 3 of algorithm implies

$$\bar{d}_k = \frac{\alpha_{k+1}}{\alpha_k} d_k \quad \text{(step 3a)}$$

maintains positivity of $\bar{d}_k$ if $d_k > 0$

- Substitute in step 1: $\bar{d}_k = d_k + \sigma_k v_{kk}^2 = d_k - \dfrac{v_{kk}^2}{\alpha_k}$

$$\frac{\alpha_{k+1}}{\alpha_k} d_k = d_k - \frac{v_{kk}^2}{\alpha_k}$$

$$\Rightarrow \alpha_{k+1} = \alpha_k - \frac{v_{kk}^2}{d_k}$$

Case 2 (contd.)

or $\alpha_k = \alpha_{k+1} + \dfrac{v_{kk}^2}{d_k}$ (Step 1a)

α_k^S are postive provided that α_k^S are computed backwards. Need α_{n+1} to initialize recursion and all v_{kk}

It is easy to initialize α_{n+1} in RLS (e.g., $\alpha_{n+1} = 1$ or c)

$$\text{if } \alpha = \underline{f}^T D \underline{f} + c \Rightarrow \alpha_{n+1} = c$$

- Also, have

$$\beta_k = -\frac{v_{kk}}{\alpha_k \bar{d}_k} = -\frac{v_{kk}}{\alpha_{k+1} d_k} \qquad \text{(Step 2a)}$$

- Key: when A is a diagonal matrix as in case 3, we can compute v_{kk} *a priori*

Case 3 : A = Diag (d_i)

- $A = Diag(d_i) \Rightarrow \underline{l}_i = \underline{e}_i =$ unit vectors
  - Note that $\underline{v}_{k+1} = \underline{v}_k - v_{kk}\underline{e}_k$
    $= \underline{v}_k$ with k^{th} element set to 0
  - So, if $$\underline{v} = \underline{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{bmatrix}, \text{ then } \underline{v}_k = \begin{bmatrix} 0 \\ . \\ v_k \\ . \\ v_n \end{bmatrix} = \underline{v}_k^{(k)}; \left(\underline{v}^{(1)} = \underline{v}; \underline{v}^{(n+1)} = \underline{0}\right)$$
  - Thus, the v_{kk} in the above case are known *a priori*

    $$\Rightarrow \underline{\bar{l}}_k = \underline{e}_k + \beta_k \underline{v}^{(k+1)} \text{ or } \bar{l}_{ik} = \beta_k v_i;\ i > k$$

Case 4 : Finally *RLS*

- Special case for RLS $\underline{v} = D\underline{f}$; $\underline{f} = L^T\underline{a}$ arbitrary since $\underline{a}$ is arbitrary
  - $\alpha = \underline{f}^T D\underline{f} + c;\ c = \text{scalar}$
  - So, here $\alpha = c + \sum_{i=1}^{n} f_i^2 d_i = \alpha_1$ and $v_k = v_{kk} = d_k f_k$
  - But from (1a): $\alpha_k = \alpha_{k+1} + f_k^2 d_k$

    $\Rightarrow \alpha_{n+1} = c$

    $\Rightarrow$ can get α_k via a backward recursion.
  - So, if $c \geq 0, \alpha_k \geq 0 \Rightarrow \bar{d}_k > 0$ (see step 3a)
  - Also, note $\beta_k = -v_k / (d_k \alpha_{k+1}) = -f_k / \alpha_{k+1}$
  - Since all $\alpha_k, \bar{d}_k$ and β_k can be computed *a priori*, we can get L (i.e., $\underline{\bar{l}}_i$) either forward or backward. But backward is preferred, since we don't have to store α_k and β_k.

Algorithm for RLS

- Due to Agee-Turner (1972); Gill, Golub, Murray & Saunders (1974)

Initialize $\alpha_{n+1} = c$

$\alpha_n = c + f_n^2 d_n$

$\bar{d}_n = \left(\alpha_{n+1} / \alpha_n\right) d_n$

For $k = n-1,\ldots,1$ DO

$$\beta_k = -f_k / \alpha_{k+1} \qquad (2a)$$

$$\alpha_k = \alpha_{k+1} + d_k f_k^2 \qquad (1a)$$

$$\bar{d}_k = d_k . \alpha_{k+1} / \alpha_k \qquad (3a)$$

$$\bar{\underline{l}}_k = \underline{l}_k + \beta_k \underline{v}^{(k+1)}$$

end DO

- Once done, we have

$$\tilde{L} = I + \begin{bmatrix} | & | & .. & | & | \\ \beta_1 \underline{v}^{(2)} & \beta_2 \underline{v}^{(3)} & .. & \beta_{n-1} \underline{v}^{(n)} & 0 \\ | & | & .. & | & | \end{bmatrix}$$

Algorithm for RLS – Details - 1

- For Least Squares, we need

$$L\left[D-\underline{v}\,\underline{v}^T/\left(c+\underline{f}^T D\underline{f}\right)\right]L^T=\bar{L}\bar{D}\bar{L}^T \quad \text{where } \underline{v}=D\underline{f}$$

$$\text{so factor }\left[D-\underline{v}\,\underline{v}^T/\left(c+\underline{f}^T D\underline{f}\right)\right]\rightarrow \tilde{L}_1\tilde{D}_1\tilde{L}_1^T \Rightarrow \bar{L}=L\tilde{L}_1$$

- But can get $\bar{L}$ directly. From above multiply by L:

$$\bar{L}=L+\begin{bmatrix} | & | & .. & | & | \\ \beta_1 L\underline{v}^{(2)} & \beta_2 L\underline{v}^{(3)} & .. & \beta_{n-1}L\underline{v}^{(n)} & 0 \\ | & | & .. & | & | \end{bmatrix}$$

$$\text{Define } L\underline{v}^{(k)}=\underline{\xi}_k,\ \text{since } l_{ii}=1\Rightarrow \underline{\xi}_k=\begin{bmatrix}0\\ \cdot\\ v_{kk}\\ \times\\ \times\end{bmatrix};\underline{\xi}_n=\begin{bmatrix}0 & \dots & 0 & v_n\end{bmatrix}$$

Algorithm for RLS - Details - 2

$$\Rightarrow \text{So, } \underline{\tilde{l}}_k = \underline{l}_k + \beta_k \underline{\xi}_{k+1}$$

$$\Rightarrow \underline{\xi}_k = L\underline{v}^{(k)} = L\underline{v}^{(k+1)} + \underline{l}_k v_{kk} \text{ since } \underline{v}^{(k+1)} = \underline{v}^k - \underline{e}_k v_{kk}$$

$$\Rightarrow \underline{\xi}_k = \underline{\xi}_{k+1} + v_{kk}\underline{l}_k = \left(1 - \beta_k v_{kk}\right)\underline{\xi}_{k+1} + \underline{\tilde{l}}_k v_{kk}$$

- Note 1: $\underline{\xi}_1 = L\underline{v}_1$

  the gain vector, $\underline{g} = P\underline{a} / \left(c + \underline{a}^T P\underline{a}\right)$

  $$= LDL^T\underline{a} / \left(f^T Df + c\right) = \underline{\xi}_1 / \alpha_1$$

  $$\Rightarrow \boxed{\underline{g} = \frac{1}{\alpha_1} \cdot \underline{\xi}_1}$$

- Note 2: Start the entire process with $L = I$, $D = 10^5 I$
  Computational load = O $(1.5n^2 + 2.5n)$

Sequential QR – Add a Measurement - 1

- Adding a new measurement $\Rightarrow$ add a new row to A
  - Suppose added new row as row 1

$$A_{k+1} = \begin{bmatrix} a_{k+1}^T \\ A_k \end{bmatrix} \text{ where } A_k = Q_k R_k,\ Q_k \to k \times n \text{ and } R_k \to n \times n$$

$$diag\left(1, Q_k^T\right) A_{k+1} = \begin{bmatrix} a_{k+1}^T \\ R_k \end{bmatrix} = H_k \Rightarrow \text{ Upper Hessenberg matrix}$$

  - For k = 4 and n = 3, H_k looks like:

$$\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & 0 \end{bmatrix}$$

  - H_k = upper Hessenberg (upper Δ + sub diagonal)

Sequential QR – Add a Measurement - 2

- Apply Givens transformations to upper triangularize H_k

$$J^T(n,n+1)\ldots J^T(2,3)J^T(1,2)H_k = R_{k+1}$$

$$\Rightarrow J_n^T \ldots J_2^T J_1^T H_k = R_{k+1}$$

$$\Rightarrow A_{k+1} = Q_{k+1}R_{k+1}$$

where $Q_{k+1} = diag(1,Q)J_1J_2\ldots J_n$

- But want to add a row k+1 at the end $\quad A_{k+1} = \begin{bmatrix} A_k \\ \underline{a}_{k+1}^T \end{bmatrix}$

- Use exchange matrix $\quad E_k = \begin{bmatrix} 0 & . & \ldots & 1 \\ 0 & .. & 1 & 0 \\ 1 & .. & . & 0 \end{bmatrix}$ k by k matrix; $E_k^2 = I_k$

and define $\bar{A} = \begin{bmatrix} \underline{a}_{k+1}^T \\ E_k A_k \end{bmatrix}, \bar{Q}_k = E_k Q_k$

$$\Rightarrow diag(1,\bar{Q}_k^T)\bar{A} = \begin{bmatrix} \underline{a}_{k+1}^T \\ R_k \end{bmatrix} = H_k$$

Sequential QR – Add a Measurement - 3

- So apply Givens transformations as before to obtain:

$$J_n^T \ldots J_2^T J_1^T H_k = R_{k+1}$$

$$J_n^T \ldots J_2^T J_1^T \underbrace{diag\left(1, \bar{Q}_k^T\right) E_{k+1} A_{k+1}}_{\bar{A}} = R_{k+1}$$

$$\Rightarrow Q_{k+1} = E_{k+1} diag\left(1, \bar{Q}_k^T\right) J_1 \ldots J_n$$

$$= \begin{bmatrix} 0 & E_k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & E_k Q_k \end{bmatrix} J_1 \ldots J_n$$

$$= \begin{bmatrix} 0 & Q_k \\ 1 & 0 \end{bmatrix} J_1 \ldots J_n$$

- O(m n) operations

Sequential QR – Drop a Measurement

- Suppose we want to delete a measurement (e.g., found to be an outlier after it was incorporated into Least Squares estimate)
  - For simplicity, assume it is 1st measurement

$$A_k = \begin{bmatrix} \underline{a}_1^T \\ A_1 \end{bmatrix}$$

  - Let $\underline{q}_1^T$ be the first row of Q
  - Compute Givens rotations $J_{m-1} \ldots J_1$

$$J_{m-1} \ldots J_1 \underline{q}_1 = \alpha \underline{e}_1 \text{ where } \alpha = \pm 1$$

$$H = J_1^T \ldots J_{m-1}^T R = \begin{bmatrix} \underline{v}^T \\ R_1 \end{bmatrix},$$

where $H =$ upper Hessenberg matrix

  - Note that $QJ_{m-1} \ldots J_1 = \begin{bmatrix} \alpha & 0 \\ 0 & Q_1 \end{bmatrix}$

Sequential QR – Add a Parameter

- It must be of this form, since it is orthogonal
- So, $A = \begin{bmatrix} \underline{a}_1^T \\ A_1 \end{bmatrix} = \left(QJ_{m-1} \ldots J_1\right) J_1^T \ldots J_{m-1}^T R$

$$= \begin{bmatrix} \alpha & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} \underline{v}^T \\ R_1 \end{bmatrix}$$

$\Rightarrow A_1 = Q_1 R_1$ is the desired Q-R factorization

- ❑ Adding a new column => increase number of parameters by 1

$$\bar{A} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \ldots & \underline{a}_n & \underline{a}_{n+1} \end{bmatrix}$$

- So, $Q^T \bar{A} = \begin{bmatrix} R\underline{w} \end{bmatrix}; \underline{w} = Q^T \underline{a}_{n+1}$

Sequential QR – Drop a Parameter - 1

- Apply $J_{n+1}^T \ldots J_{m-1}^T w = \begin{bmatrix} \times \\ \times \\ \times \\ 0 \\ . \\ 0 \end{bmatrix} \leftarrow n+1 \quad \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ 0 & 0 & 0 & \times \end{bmatrix}$

  $\Rightarrow$ does not change R and $Q = QJ_{m-1} \ldots J_{n+1}$

  $\Rightarrow$ computational load: O(mn) operations

❑ Delete column $k \Rightarrow$ remove x_k (or k^{th} factor) $\Rightarrow$ reduce the number of parameters by 1

- $\bar{A} = [\underline{a}_1 \quad \ldots \quad \underline{a}_{k-1} \quad \underline{a}_{k+1} \quad \ldots \quad \underline{a}_n]$

Sequential QR – Drop a Parameter - 2

- Write

$$Q^T A = \begin{array}{c|c} \begin{array}{ccc} k-1 & 1 & n-k \end{array} & \\ \begin{bmatrix} R_{11} & v & R_{12} \\ 0 & r_{kk} & \underline{w}^T \\ 0 & 0 & R_{33} \end{bmatrix} & \begin{array}{c} k-1 \\ 1 \\ m-k \end{array} \end{array}$$

$$Q^T \bar{A} = \begin{bmatrix} R_{11} & R_{12} \\ 0 & \underline{w}^T \\ 0 & R_{33} \end{bmatrix} \begin{array}{c} k-1 \\ 1 \\ m-k \end{array} = H \Rightarrow \text{Upper Hessenberg from columns } (k+1) \text{ to } n$$

- Consider m = 5, n = 4, k = 2

$$\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow J_{n-1}^T \ldots J_{k+1}^T J_k^T H = R_1$$

$$\Rightarrow Q = Q J_k J_{k+1} \ldots J_{n-1}$$

Computational load: O(n^2)

- Zero out unwanted sub diagonals $h_{k+1,k} \ldots h_{n,n-1}$

Application to Kalman Filtering - 1

- Consider the LTI dynamic model of a stochastic system:

Dynamics: $\underline{x}_{k+1} = \Phi \underline{x}_k + E \underline{w}_k$

Measurement: $\underline{y}_k = H \underline{x}_k + \underline{v}_k$

- Note that Φ, G, and H can be time-varying
- But, we will assume that they are time-invariant for simplicity of notation
- $\{\underline{w}_k\}$ process noise sequence. Assumed to be zero-mean white, Gaussian noise sequence with covariance matrix, W_d
- $\{\underline{v}_k\}$ measurement noise sequence. Assumed to be zero-mean, white Gaussian noise sequence with covariance matrix R
- Without loss of generality, assume that W_d and R are diagonal

Application to Kalman Filtering - 2

- **If not:**

Form $W_d = L_w D_w L_w^T$

$$\text{and define:}\quad \left.\begin{aligned} &\text{new } E = E L_w \\ &\text{new } W_d = D_w \\ &\text{new } \underline{w}_k = L_w^{-1}\underline{w}_k \end{aligned}\right\} \text{called whitening of process noise}$$

- **Similarly, if R is not diagonal**

Form $R = L_r D_r L_r^T$

$$\text{and define:}\quad \left.\begin{aligned} &\text{new } \underline{y}_k = L_r^{-1}\underline{y}_k \\ &\text{new } H = L_r^{-1} H \\ &\text{new } R = D_r \\ &\text{new } \underline{v}_k = L_r^{-1}\underline{v}_k \end{aligned}\right\} \text{called whitening of observation errors}$$

- Kalman filter provides the minimum mean-square error (MMSE) estimate (also called maximum a posteriori (MAP) estimate).
- If the initial state $\underline{x}_0$ is Gaussian with mean $\underline{\bar{x}}_0$ and covariance matrix P_0, define:
- $\hat{\underline{x}}_{k/k}$ = best estimate of $\underline{x}_k$ based on measurements $\{\underline{y}_1, \ldots \underline{y}_k\}$
- $\hat{\underline{x}}_{k+1/k}$ = best estimate of $\underline{x}_{k+1}$ based on measurements $\{\underline{y}_1, \ldots \underline{y}_k\}$

Kalman Filter Equations

- The Kalman filter equations are:

$$\underline{\hat{x}}_{k+1/k} = \Phi \underline{\hat{x}}_{k/k} \quad \text{.... (PROPAGATE or PREDICTION STEP)}$$

$$\underline{\hat{x}}_{k/k} = \underline{\hat{x}}_{k/k-1} + \underbrace{G_k}_{\substack{\text{Kalman}\\ \text{Gain}}} \underbrace{(y_k - H\underline{\hat{x}}_{k/k-1})}_{\text{innovation}} \quad \text{.... (UPDATE STEP)}$$

- Different filter algorithms differ in the way they compute the Kalman gains $\{G_k\}$
  - Conventional Kalman filter:

$$G_k = P_{k/k-1} H^T \left(H P_{k/k-1} H^T + R \right)^{-1}$$

– update step: $P_{k/k} = (1 - G_k H) P_{k/k}$

– propagate step: $P_{k+1/k} = \Phi P_{k/k} \Phi^T + E W_d E^T$

where $P_{k/k} = E\left[\left(\underline{x}_k - \underline{\hat{x}}_{k/k} \right) \left(\underline{x}_k - \underline{\hat{x}}_{k/k} \right)^T \right]$

$$P_{k+1/k} = E\left[\left(\underline{x}_{k+1} - \underline{\hat{x}}_{k+1/k} \right) \left(\underline{x}_{k+1} - \underline{\hat{x}}_{k+1/k} \right)^T \right]$$

Round-off Error Problems

Remarks on the conventional Kalman filter:

- The update step can be implemented recursively one measurement at a time. This is because:

$$P_{k/k}^{-1} = P_{k/k-1}^{-1} + H^T R^{-1} H = P_{k/k-1}^{-1} + \sum_{i=1}^{m} r_i \underline{h}_i \underline{h}_i^T$$

$$\underline{h}_i^T = i^{th} \text{ row of } H$$

$$r_i = i^{th} \text{diagonal element of } R$$

$$m = \text{number of measurements} = \#\text{of rows in } H$$

- Need to compute only $n(n+1)/2$ elements of P, since P is symmetric.
- Could get negative diagonal elements in P $\left(\text{if } r_i \text{ and/or } w_{di} \text{ are small}\right)$.

Joseph's stabilized measurement update:

$$P_{k/k} = \left(1 - G_k H\right) P_{k/k-1} \left(1 - G_k H\right)^T + G_k R G_k^T$$

$$= \left(1 - G_k H\right) P_{k/k-1} \left(1 - G_k H\right)^T + \sum_{i=1}^{m} r_i \underline{g}_{ki} \underline{g}_{ki}^T, \quad \underline{g}_{ki} = i^{th} \text{column of } G_k$$

Solution Approaches

❑ Conventional Kalman filter with lower bounding:

- Compute:

$$\bar{P} = (1 - G_k H) P_{k/k-1}$$

$$(P_{k/k})_{jj} = \max\left(\bar{P}_{jj}, \sigma^2_{\min,j}\right); \quad j = 1, 2, \ldots, n$$

$$(P_{k/k})_{ij} = \begin{cases} \bar{P}_{ij} \text{ if } \bar{P}^2_{ij} < M_{ij} \\ \text{sign}\left(\bar{P}_{ij}\right)\sqrt{M_{ij}} \text{ otherwise} \end{cases}$$

$$\text{where } M_{ij} = \rho^2_{\min} (P_{k/k})_{ii} (P_{k/k})_{jj}$$

– Selection of $\rho_{\min}$ and $\sigma_{\min}$ is an art

– Does not guarantee positive definiteness of $P_{k/k}$

(see Kerr, IEEE T-AES, Nov. 1990)

❑ LDL^T Factorization:

- We will present the **algorithm in two steps:**

1. update step
2. propagation step

Kalman Filter via LDL^T Updates

❑ LDL^T Factorization: Measurement update step

- **Trivial application of previous Least-squares update algorithm**
  - We know that $P_{k/k}^{-1} = P_{k/k-1}^{-1} + \sum_{i=1}^{m} r_i \underline{h}_i \underline{h}_i^T$
  - So, implement via:

    DO $i = 1, m$

    call previous Agee-Tumer algorithm with $(r_i, \underline{h}_i)$ and current L and $\underline{d}$

    end DO

- **Factorization problem associated with propagation step:**
  - Recall that: $P_{k+1/k} = \Phi P_{k/k} \Phi^T + E W_d E^T$
  - Problem: Given $P_{k/k} = LDL^T$, we seek $\bar{L}, \bar{D}$ such that $P_{k+1/k} = \bar{L}\bar{D}\bar{L}^T$

LDL^T Update Methods for Kalman Filters

- There are basically 3 methods to obtain $\bar{L}, \bar{D}$ of $P_{k+1/k}$:
  - **Method 1**
    - Let $EW_dE^T = L_eD_eL_e^T$ be the factorization of EW_dE^T
    - Further, let $\underline{l}_i = \underline{\gamma}_i$ for $i = 1, 2, \ldots, n,$ where $\underline{l}_i =$ column i of L
    - Then:

$$P_{k+1/k} = L_eD_eL_e^T + \underbrace{\sum_{i=1}^{m} d_i \underline{\gamma}_i \underline{\gamma}_i^T}_{\text{modifying rank-one corrections}}$$

    - So, the algorithm is:

      DO $i = 1, n$

      call case 1 of general algorithm with $\left(d_i, \underline{\gamma}_i\right)$

      end DO

    - Probem: EW_dE^T is typically positive semi-definite

Gram-Schmidt for Propagation Step - 1

Method 2: Weighted Gram-Schmidt

- Recall that $P_{k+1/k} = [\Phi L \quad E]\begin{bmatrix} D & 0 \\ 0 & W_d \end{bmatrix}[\Phi L \quad E]^T = A^T \tilde{D} A = \bar{L}\bar{D}\bar{L}^T$

  where $A = \begin{bmatrix} L^T \Phi^T \\ E^T \end{bmatrix}$, $\tilde{D} = \begin{bmatrix} D & 0 \\ 0 & W_d \end{bmatrix}$

  – Obtain $\bar{L}\bar{D}\bar{L}^T$ via parallel weighted Gram-Schmidt orthogonalization procedure.

  – Idea: Obtain a set of orthogonal directions $(\underline{q}_1, \underline{q}_2, ..., \underline{q}_n)$ where $\underline{q}_i \in R^{n+n_w}$

  $$A = \begin{bmatrix} L^T \Phi^T \\ E^T \end{bmatrix} = (\underline{a}_1, \underline{a}_2, ..., \underline{a}_n) = [\underline{q}_1, \underline{q}_2, ..., \underline{q}_n]\bar{L}^T = Q\bar{L}^T = QR$$

  where $\bar{L}$ = unit lower Δ, R = unit upper Δ

  and $\underline{q}_i^T \tilde{D} \underline{q}_j = 0 \ \forall \ i \neq j \ \Rightarrow \ \{\underline{q}_i\}$ are $\tilde{D}$-orthogonal.

  – Once $\underline{q}_i^T$ are known, we can obtain $\bar{D} = diag(\bar{d}_1, \bar{d}_2, ..., \bar{d}_n)$ via:

  $$\bar{d}_i = \underline{q}_i^T \tilde{D} \underline{q}_i; \ i = 1, 2, .., n$$

  – After D-orthogonalization, $P_{k+1/k} = A^T \tilde{D} A = \bar{L} Q^T \tilde{D} Q \bar{L}^T = \bar{L}\bar{D}\bar{L}^T$

Gram-Schmidt for Propagation Step - 2

Method 2 cont…

- The algorithm for $\tilde{D}$-orthogonalization of A is a minor variation of parallel Gram-Schmidt.
  - The matrix A is replaced by Q
  - Algorithm:

For $k = 1, 2, \ldots, n$ DO

$$\bar{d}_k = \underline{a}_k^T \tilde{D} \underline{a}_k$$

$$r_{kk} = 1 \quad \rightarrow \; l_{kk} = 1$$

For $j = k+1, \ldots, n$ DO

$$r_{kj} = \frac{\underline{a}_j^T \tilde{D} \underline{a}_k}{\bar{d}_k} \quad \rightarrow \text{ obtained } l_{jk}$$

$$\underline{a}_j \leftarrow \underline{a}_j - r_{kj} \underline{a}_k$$

end DO

end DO

Method 3: Householder or Givens Transformations

- Recall that we can find transformation Q such that

$$\tilde{D}^{1/2} A = Q \bar{D}^{1/2} \bar{L}^T$$

$$\rightarrow P_{k+1/k} = \mathrm{A}^T \tilde{\mathrm{D}} \mathrm{A} = \bar{\mathrm{L}} \bar{\mathrm{D}} \bar{\mathrm{L}}^T$$

Other Applications of Sqrt Updates

- **Other applications of square-root updates**
  - Probabilistic data association filter (PDAF) to track targets in clutter – additional m rank-one corrections in the measurement update equations
  - Quasi-Newton methods in non-linear programming
    - rank-two or rank-three corrections

- **References:**

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3.) "Special Issue on Factorized Estimation Applications, IEEE Trans. On Automatic Control, Dec. 1990.
4.) V. Raghavan, K.R. Pattipati and Y. Bar-Shalom, "Efficient L-D Factorization Algorithms for PDA, IMM, and IMM-PDA Filters," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 29, October 1993, pp. 1297-1310.

Summary

- Recursive (sequential) Least Squares
- Sequential LDL^T Factorization updates
- Sequential QR updates
- Application to Kalman filtering