Lectures 7-8

Classical SISO (Continuous-time) Control Design Methods

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Digital Control System Design Methods: Classical and Modern

Performance Criteria and the Design Process

- Tools for Control Design and Analysis
- Loop shaping: Trade-offs and issues
- Design Methods
  - Lag compensator design
  - Lead compensator design
  - Lead-Lag Design
  - PID controller design
    - Different PID structures
    - Integral windup protection
    - PID parameter selection rules
  - IMC design (Shaping S, T or $Q = HS$)
  - Weighted sensitivity and IMC ("Model Matching")
    - Co-prime factorization via state space
    - Design for unstable and non-minimum phase plants

Tools for Control Design and Analysis

- Bode plots
  - $G(s)$ vs. $\tilde{G}(z)$, $LG_{ain}(z)\big|_{z = e^{j\omega h}}$, $S(z)$, $T(z)$
- State variable analysis
- Computer programs
  - ss2tf, c2d, bode, margin, lsim or your own control simulation program, rlocus, nyquist...
- Root locus
- Nyquist
- Nichols

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With a system model and performance specifications in hand, we are now ready to design a digital control algorithm.

=> But first, let's review classical series compensation design methods used for continuous time systems.

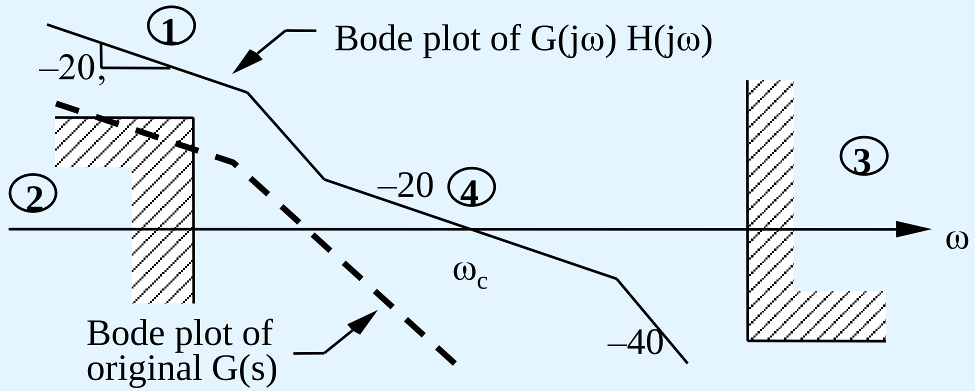
"Can't know where you're going if you don't know where you've been!"

Loop Shaping : Trade-offs and Issues

(1) Graphical methods to pick H(s) via Bode plot modifications.

(2) s-plane methods to pick H(s) via root locus shaping.

- These are trial and error methods since frequency domain (s-plane) measures are not 1:1 with time-domain measures (e.g., step response), especially for higher order systems.
- Bode plot design



① At $\omega \to 0$, $G(s)H(s) \to K_v/s$, i.e., $K_v = \lim sKG(s)H(s)$ as $s \to 0$. Restrictions on ss tracking error to a ramp input will set DC gain of GH (recall ss error to ramp input command $r(t) = \beta t$ is β/K_v).

② Since $\left|\frac{e(s)}{r(s)}\right| = \frac{1}{|1+G(s)H(s)|}$ restrictions on ss accuracy over mid-frequency range will give lower bound on | GH | (e.g., for < 2% relative error over $[0,\bar{\omega}]$, $|GH| > 50$ for $\omega < \bar{\omega}$).

③ At high frequencies, for noise rejection we want | G(s) H(s) | to be small (e.g., $|G(s)H(s)| < 0.01$, $\omega > \omega_{max}$).

④ May have restrictions on ω_c ~ bandwidth. Also may wish $\phi_m > 45^o$ (or as large as possible) via stability criterion (viz, phase curve of GH).

Classical Design Techniques

- Bode plot approach:

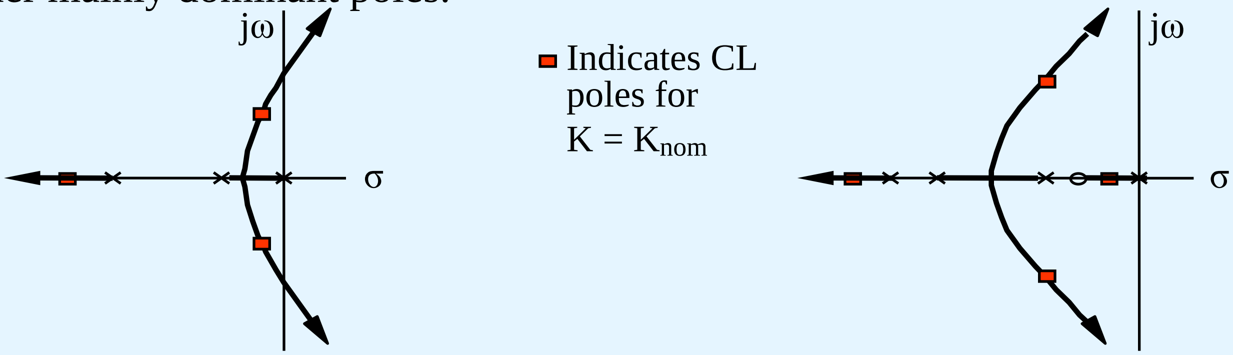
  Sketch Bode plot of G(s) and then add in gain plus poles and zeros of H(s) to bend/shape G(s)H(s) to meet specs.

  => "Create a fair stretch of –20dB/decade slope in the crossover region by choice of H(s) with $\phi_m \sim 45^o$ ".

  "fair stretch" ~ ± 1 octave $[\omega_c/2, 2\omega_c]$ or greater

  - Must next evaluate CL poles, zeros, time-response, etc.
- Root locus approach:

  Bend and shape root locus (RL) of G(s) by adding (real) poles and zeros so that the RL passes through "desirable" regions in the s-plane.Then pick gain of H to place poles. Consider mainly dominant poles.



a) Root locus of CL poles of uncompensated system [i.e., H(s) = K·1]

b) Root locus of CL poles of compensated system $H(s) = K\frac{1+s/a}{1+s/b}$

- Must next evaluate ϕ_m, bandwidth, time-response, etc.
- Useful approximation for 2nd order continuous system (ϕ_m in deg) $\zeta \sim (1+\phi_m/190^o)\phi_m/130^o$

Rule of Thumb

"create a fair stretch of -20 db slope in the crossover region by choice of K & H(s) with $\phi_m \sim 45^\circ$ "

==> want the system to act as a first order system near cross-over

==> $\phi_m \sim 45^o$ for stability & relative stability.

Once design is done, you must evaluate poles and zeroes, step response, etc. since they are not immediately evident from the frequency response plot.

TYPES OF COMPENSATION : Bode Plot of G(s) alone will usually not satisfy requirements

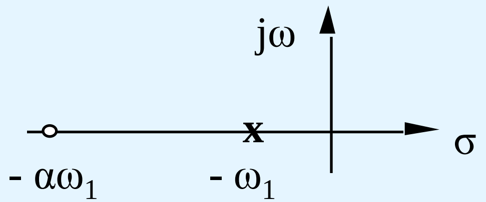
1. Pure Gain (or Gain compensation) K (H(s) = 1) fairly limited
2. Lag network $H(s) = \dfrac{1 + s/\alpha\omega_1}{1 + s/\omega_1}$; $\alpha > 1$ (but rarely ever > 20)

Typically $1 < \alpha < 20$

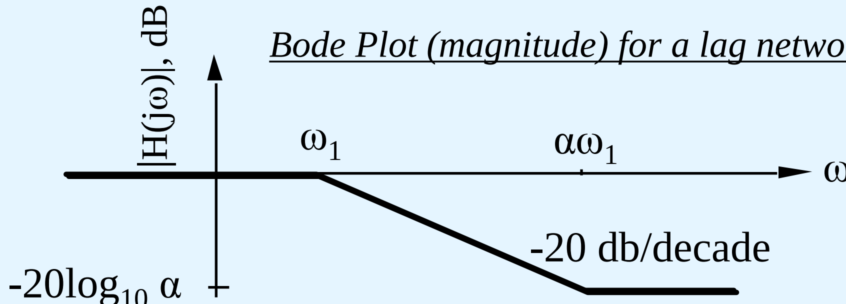
$$H(s) = \frac{1 + s/\alpha\omega_1}{1 + s/\omega_1} = \frac{1}{\alpha}\cdot\frac{s+\alpha\omega_1}{s+\omega_1} = \frac{1}{\alpha} + \frac{(1 - 1/\alpha)\,\omega_1}{(s+\omega_1)}$$

for $\alpha\omega_1 >> 1$ & $\omega_1 \approx 0$ $\rightarrow H(s) = k_2 + k_1/s$

$\approx 1/\alpha + \omega_1/s$ *PI Controller*

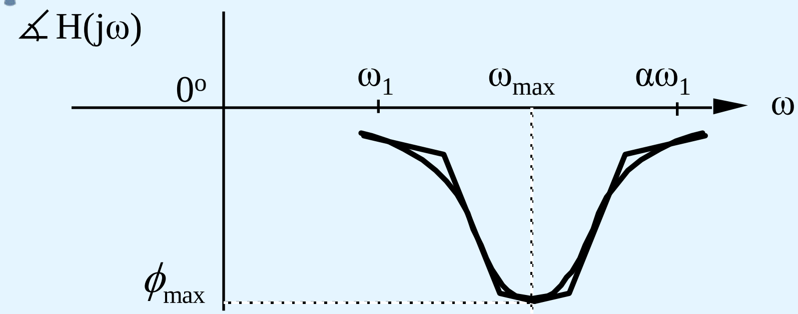


Zero/Pole pattern for a lag network



Bode Plot (magnitude) for a lag network

Lag Network - 1



$$\omega_{max} = \sqrt{\alpha}\cdot\omega_1$$

$$\phi_{max} = -\sin^{-1}\frac{\alpha-1}{\alpha+1}$$

Bode Plot (phase) for a lag network

Let us look at the phase

$$\phi = \tan^{-1}(\omega/\alpha\omega_1) - \tan^{-1}(\omega/\omega_1) = \tan^{-1}\frac{\omega/\alpha\omega_1 - \omega/\omega_1}{1+\omega^2/\alpha\omega_1^2} = \tan^{-1}\left[\frac{\dfrac{\omega}{\omega_1}\dfrac{(1-\alpha)}{\alpha}\alpha\omega_1^2}{(\omega_1^2\alpha+\omega^2)}\right]$$

$$= \tan^{-1}\frac{\omega\omega_1(1-\alpha)}{\omega_1^2\alpha+\omega^2} = \tan^{-1}\frac{1-\alpha}{\omega/\omega_1 + \alpha\omega_1/\omega}$$

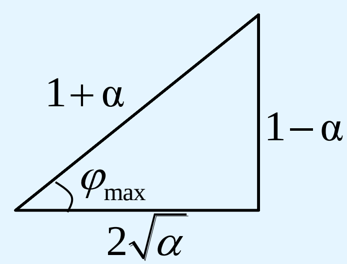
$\tan(\phi)$ is a monotonic function of ϕ for all $\phi \in (0^\circ, 90^\circ)$

$\tan(\phi)$ is a maximum ϕ_{max}

$$\Rightarrow \frac{d}{d\omega}\frac{1-\alpha}{(\omega/\omega_1+\alpha\omega_1/\omega)} = 0 \;\Rightarrow\; 1/\omega_1 - \alpha\omega_1/\omega^2 \;\Rightarrow\; \omega = \sqrt{\alpha}\,\omega_1$$

$$\phi_{max} = \tan^{-1}\frac{1-\alpha}{2\sqrt{\alpha}}$$

$$= \sin^{-1}\frac{1-\alpha}{1+\alpha} = -\sin^{-1}\frac{\alpha-1}{\alpha+1}$$



Lag Network - 2

- Easily built via an RC network:

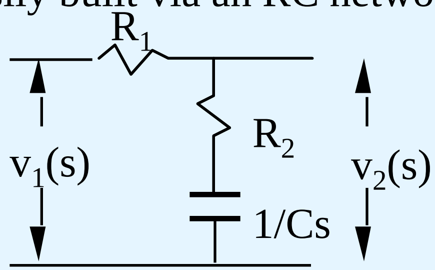


Figure III.8 Lag Network

$$\Longrightarrow \frac{v_1(s)-v_2(s)}{R_1}=\frac{v_2(s)}{R_2+1/cs}$$

$$\Longrightarrow \frac{v_1(s)}{R_1}=v_2(s)\left[\frac{1}{R_1}+\frac{1}{R_2+1/cs}\right]$$

The transfer function is:

$$\frac{v_2(s)}{v_1(s)}=\frac{1}{R_1\left[\frac{1}{R_1}+\frac{1}{R_2+1/cs}\right]}=\frac{1}{R_1\left[\frac{1}{R_1}+\frac{cs}{1+R_2cs}\right]}=\frac{1+R_2cs}{1+(R_1+R_2)cs}\;\Rightarrow\;\alpha\omega_1=\frac{1}{R_2c}$$

$$\text{Therefore, }\;\omega_1=\frac{1}{(R_1+R_2)c};\quad \alpha=\frac{R_1+R_2}{R_1}$$

Basic equations for a lag network are:

$$\phi_{max}=-\sin^{-1}\left[\frac{\alpha-1}{\alpha+1}\right]\quad\Rightarrow\quad -\alpha\sin\phi_{max}-\sin\phi_{max}=\alpha-1$$

$$\omega_{max}=\sqrt{\alpha}\;\omega_1$$

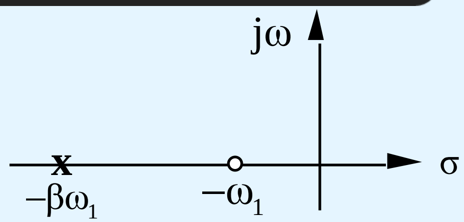
$$\omega_1=1/(R_1+R_2)C\quad\Rightarrow\quad \alpha=\frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}$$

$$\alpha=1+R_2/R_1$$

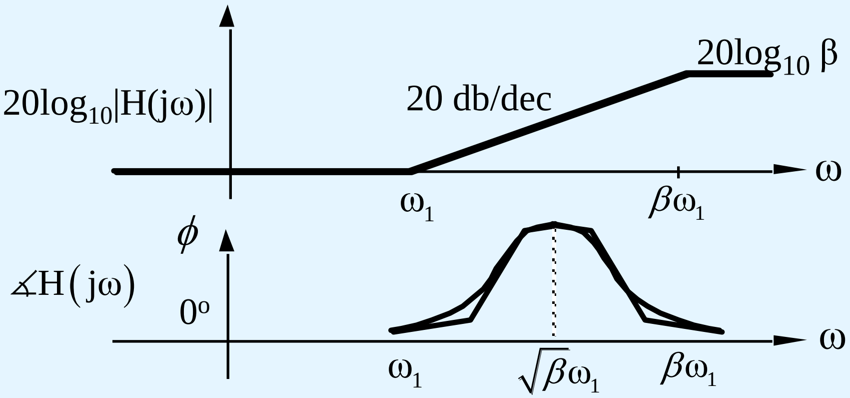
Lead Network

3. Lead Network

$$H(s) = \frac{1 + s/\omega_1}{1 + s/\beta\omega_1} \qquad \beta > 1$$



Zero/Pole pattern for a lead network



Bode Plot for a lead network.

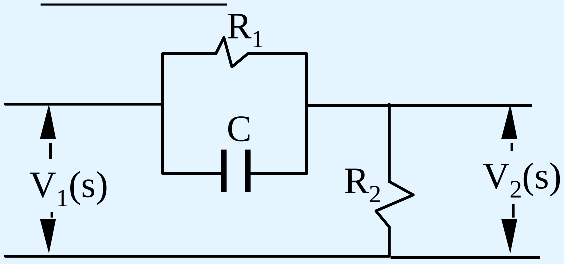
$$\omega_{max} = \sqrt{\beta} \cdot \omega_1$$

$$\phi_{max} = +\sin^{-1}\frac{\beta - 1}{\beta + 1}$$

$$\Rightarrow \beta = \frac{1 + \sin\phi_{max}}{1 - \sin\phi_{max}}$$

Typically $3 < \beta < 30$

This is a generalization of PD control. It can be built via:



Lead Network

$$H(s) = \frac{1 + s/\beta\omega_1 + s(\beta - 1)/\beta\omega_1}{1 + s/\beta\omega_1}$$

$$= 1 + \frac{T_2 s}{1 + T_2 s/N}$$

$$T_2 = \frac{(\beta - 1)}{\beta\omega_1}; N = \beta - 1$$

PD

Lag-Lead Network - 1

To see this, we note:

$$V_2(s) = \frac{1 + sR_1C}{\left[1 + \frac{R_2R_1Cs}{R_1 + R_2}\right]} \cdot \frac{R_2}{R_1 + R_2}$$

where $\omega_1 = 1/R_1C$

$$\beta = \frac{R_1 + R_2}{R_2}; N = \frac{R_1}{R_2}$$

Note: Low frequency attenuation

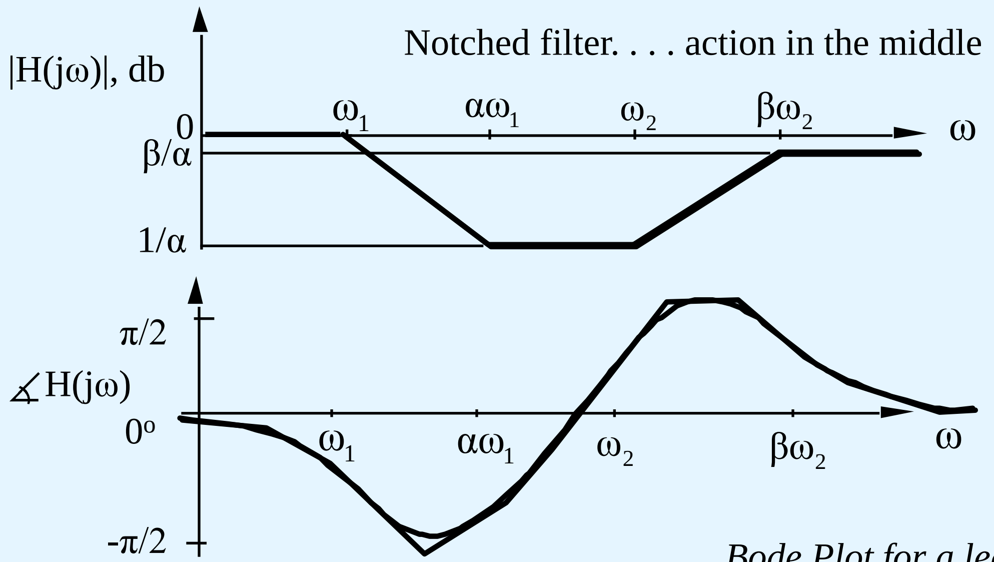
At low frequencies the gain is $1/\beta$ and at high frequencies it is unity. We need to use an operational amplifier of gain β to recover the gain.

4. Lag-Lead Network

$$H(s) = \frac{(1 + s/\alpha\omega_1)(1 + s/\omega_2)}{(1 + s/\omega_1)(1 + s/\beta\omega_2)}; \quad \alpha, \beta > 1; \quad \alpha\omega_1 < \omega_2$$

Note that: $|H(j\omega)| = \beta/\alpha$ as $\omega \to \infty$

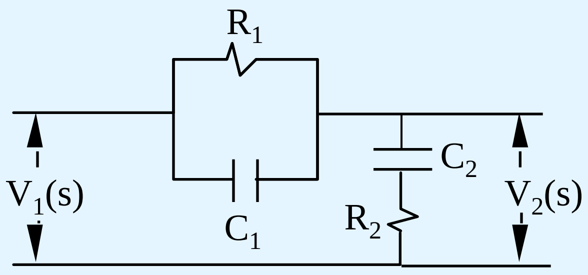
Notched filter. . . . action in the middle



Bode Plot for a lead-lag Network

Lag-Lead Network - 2

If α = β the circuit can be built via:



Lead-Lag network for α = β

$$\frac{V_2(s)}{V_1(s)} = \frac{(s + 1/C_1R_1)(s + 1/C_2R_2)}{s^2 + (1/C_1R_1 + 1/C_2R_2 + 1/C_1R_2)s + 1/C_1R_1C_2R_2}$$

This is the most flexible compensation - but requires greatest design effort.

Methods of Series Compensation

Goal: Create a fair stretch of -20dB in crossover region with $\phi_m > 45^o$.

==> Have a well behaved phase curve !

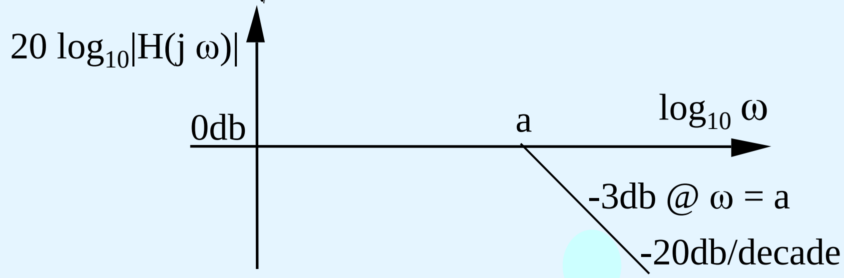
Recall Bode approx. to $H(j\omega) = \dfrac{1}{1 + j\omega/a}$; $|H(j\omega)| = \dfrac{1}{\sqrt{1 + \omega^2/a^2}}$

A) Magnitude $\omega < a \quad |H(j\omega)| \approx 1$

$\Rightarrow 20 \log_{10}|H(j\omega)| = 0$ db

$\omega > a \quad |H(j\omega)| \approx a/\omega$

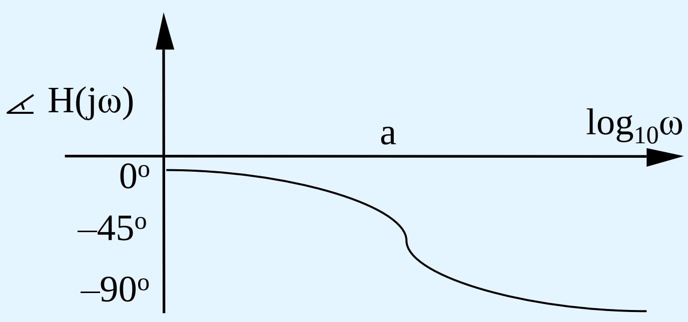
$\Rightarrow -20 \log_{10}\omega + 20 \log_{10} a$



Magnitude plot of H(jω)

B) Phase: $\phi = -\tan^{-1}\omega/a : \omega \ll a \Rightarrow \phi \approx -\omega/a$ rads ; $\omega \gg a \Rightarrow \phi \approx -\pi/a + a/\omega$

based on the equality $\tan^{-1}\omega/a + \tan^{-1} a/\omega = \pi/2 \;\forall\; a \;\&\; \omega$



Phase plot of H(jω)

Useful numbers:

$\tan^{-1} 1/2 = 26.5^\circ \approx \dfrac{26.5}{180}\pi$ rads

$\tan^{-1} 1/\sqrt{3} = \pi/6 \text{ rad} = 30^\circ$

$\tan^{-1} 3/4 = 36.9^\circ$

$\tan^{-1} 2 = 63.5^\circ$

$\tan^{-1}\sqrt{3} = 60^\circ$

$\tan^{-1} 4/3 = 53.1^\circ$

$\tan^{-1} 1 = 45^\circ$

Let us consider all the possible methods for series compensation.

Gain Compensation

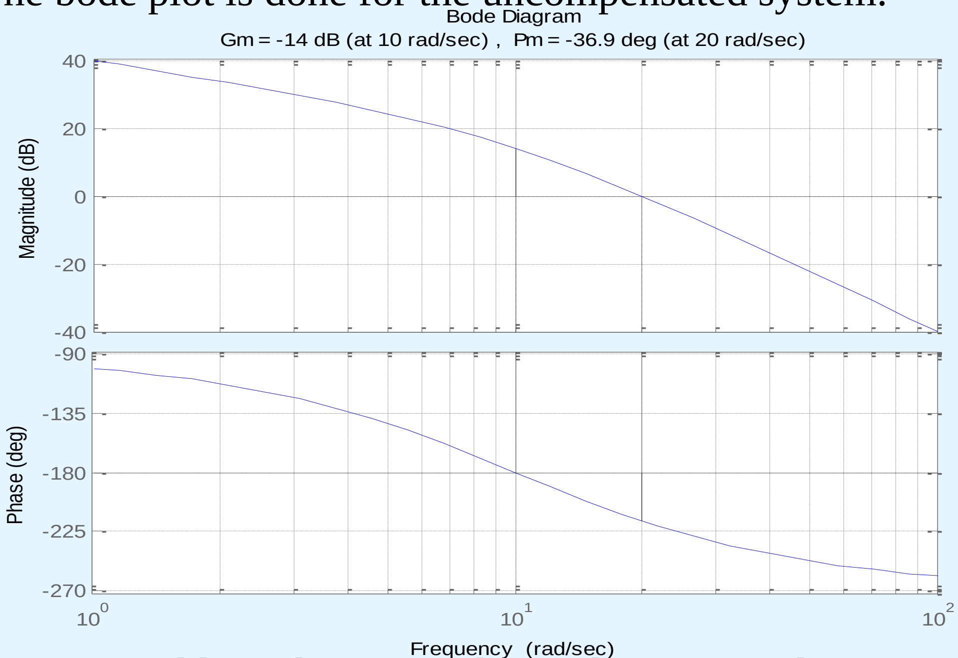
- When we use gain compensation we:
  1. Can increase gain to get desired K_v & LF accuracy
  2. Lower gain to get desired ϕ_m as phase ↓ with increase in ω ↑ (i.e. achieve desired degree of stability)

  Rarely can we satisfy both with only K.

EXAMPLE: Consider the system: $G(s) = \dfrac{100}{s(1+s/10)^2}$

Design specification: $\phi_m = 45^o$
Let us assume that K = 1.

(1) The bode plot is done for the uncompensated system.



Bode plots of uncompensated system

System is unstable with K = 1. Obviously, we must lower magnitude curve. We want gain crossover at about 4 rads/sec. Why?

Gain Compensation (Cont'd)

Let us look at $\measuredangle G(j\omega)$: Since we want $\phi_m = 45^o$ (design specification)

$$\measuredangle G(j\omega) = -\pi/2 - 2\tan^{-1}\omega_c/10 = -\pi/2 - 2\,\omega_c/10 = -3\pi/4$$

$$\Longrightarrow \omega_c = 5\pi/4 \approx 4 \text{ rads/sec} \quad (3.8 - 3.9 \text{ rads/sec})$$

Then by definition of ω_c , with $\omega_c = 4$ rads/sec

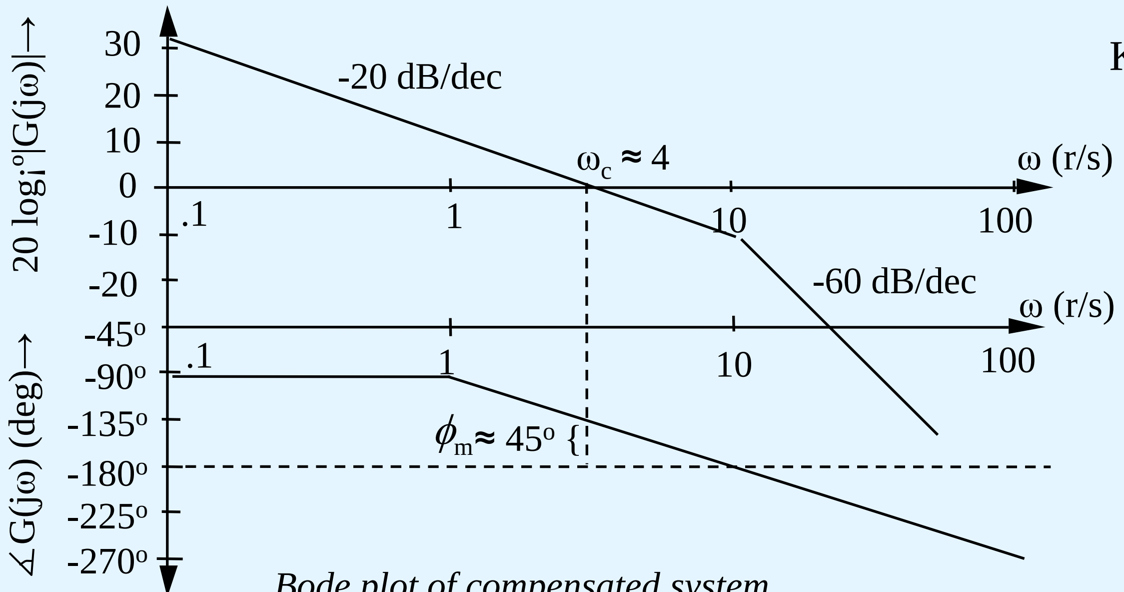
$$KG(j4) = 1 \quad \text{(corresponding to 0 db)}$$

$$|KG(j4)| = 1 \Longrightarrow K\cdot\frac{100}{\omega\cdot 1} = 1 \Longrightarrow K = \frac{4}{100} = 1/25$$

where we have used the Bode "straight line" approximation to compute $|KG(j\omega)|$

i.e. $|1+j\omega_c/10|^2 \approx 1$, since $\omega_c < 10$

Looking at the bode plot of the compensated system.



Bode plot of compensated system

$KG(s)$ = compensated system:

$$\frac{4}{s(1+s/10)^2} \Rightarrow K_v = 4$$

Gain Compensation (Cont'd)

So, get 25% error to a ramp input. However, HF attenuation is OK.

Problems with gain compensation: (1) must have a frequency where $\varphi_m \approx 45^o$

(2) Destroys LF accuracy.

Wouldn't it be nice if we could modify the magnitude plot as this would leave $K_v = 100$, its original value. Recall that the phase shift at the crossover frequency and therefore ϕ_m, depends only on the magnitude plot one decade above and below ω_c . This is precisely what a lag compensation does !!

Lag Compensation - 1

(a) used to lower cross over frequency by reducing gain without changing very low frequency gain $\Rightarrow$ can get good steady state accuracy!

(b) Easy to do on a Bode diagram, since phase $\measuredangle$ add.

(c) Must already have $\measuredangle G(j\omega) = -135^o$ in intended crossover region (since lag compensation lowers phase)

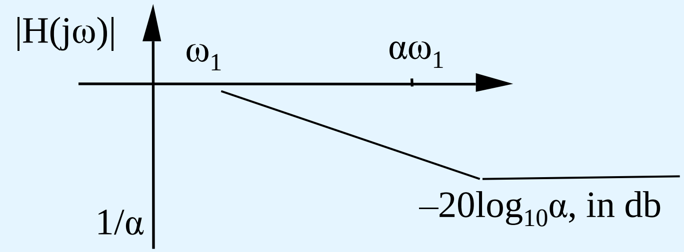
Recall that the transfer function of a lag compensator is given by:

$$H(s) = \frac{1 + s/\alpha\omega_1}{1 + s/\omega_1} \quad (\alpha > 1)$$

Since lag Network puts in phase lag, we better have ω_1 and $\alpha\omega_1$ well below Xover frequency so as not to destroy things at $\alpha\omega_1$; but not too far away. One “rule of thumb” is to choose:

$$\alpha\omega_1 = \frac{\omega_c}{10}$$

Recall that $\left|H(j\omega)\right| = \frac{\sqrt{1+(\omega/\alpha\omega_1)^2}}{\sqrt{1+(\omega/\omega_1)^2}} \Rightarrow$



Magnitude plot for a lag compensator

$$\measuredangle H(j\omega) \text{ at } \omega=\omega_c \text{ is } \tan^{-1}\frac{\omega_c}{\alpha\omega_1} - \tan^{-1}\frac{\omega}{\omega_1} = \frac{\pi}{2} - \frac{\alpha\omega_1}{\omega_c} - \frac{\pi}{2} + \frac{\omega_1}{\omega_c} = \frac{\omega_1(1-\alpha)}{\omega_c} = -\frac{\omega_1(\alpha-1)}{\omega_c}$$

- For the previous example, amount of gain reduction ("attenuation") = $1/\alpha$

$\Rightarrow \alpha = 25$ (Recall that $K_v = 4$, 25% error to a ramp input)

Lag Compensation - 2

using the rule $\alpha\omega_1 = \omega_c /10$ ==> $\omega_1 = 0.4/25 = 0.016$ rads/sec

$\alpha\omega_1 = 0.4$ rads/sec

This is a 25 : 1 ratio (a little on the high side)

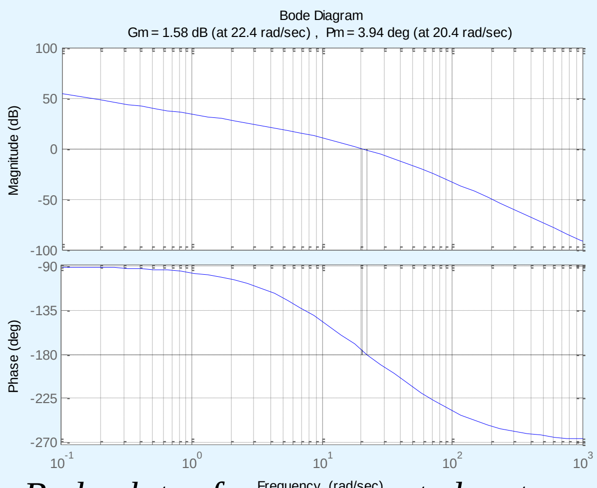
Example:

$$G(s) = \frac{5}{s(1 + s/10)(1 + s/50)}$$ ==> $K_v = 5$ with H(s) = 1, K = 1, 20% error to ramp.

Specifications: $K_v \geq 50$ (2% relative error to a ramp) and $\phi_m > 45^o$ & no restriction on ω_c

(1) Find K to meet LF requirements: $K_v = \lim_{s\to 0} sG(s) = \frac{5K}{(1)(1)} \geq 50 \Rightarrow K \geq 10$

(2) Sketch Bode plot for KG(s) to check if compensation is necessary & type needed, after selecting K to meet LF requirements. Usually this destroys stability and ϕ_m is not OK.



Bode plots of uncompensated system

Lag Compensation - 3

Now we are nearly unstable, $\phi_m \approx 4^o$. Therefore, we want to reduce gain near crossover.

(3) Find frequency at which $\phi_m = 45^o$

Let us see what the crossover should be by setting $\measuredangle KG(j\omega_c) = -3\pi/4(-135^o)$ for the 45^o desired phase margin.

Using once again the Bode approximation (for $\omega < 10$), we have:

$$\measuredangle KG(j\omega_c) = -\pi/2 - \omega_c/10 - \omega_c/50 = -3\pi/4 \quad (\text{neglecting } \measuredangle H(j\omega))$$

$$\Longrightarrow 6\omega_c/50 = \pi/4 \Longrightarrow \omega_c = 25\pi/12 \approx 6.5 \text{ rads/sec.}$$

Use a lag network to lower gain so that $\omega_c \approx 6.5$ rads/sec. However, it is better to set $\omega_c \approx 6$. This will anticipate a few degrees of lag from $\measuredangle H(j\omega)$.

Recalling a lag network: $H(s) = \dfrac{1 + s/\alpha\omega_1}{1 + s/\omega_1}$

(4) Find α to get desired ω_c .

$$\left|KGH\right|_{\omega=\omega_c=6\text{ r/s}} = 1 \quad \text{or} \quad 20\log_{10}\left|KGH\right|_{\omega=\omega_c=6\text{ r/s}} = 0\text{dB}$$

$$\left|KG(j\omega)H(j\omega)\right| = \left|\frac{50}{j\omega(1+j\omega/10)(1+j\omega/50)} \cdot \frac{1+j\omega/\alpha\omega_1}{(1+j\omega/\omega_1)}\right|$$

← ~ $1/\alpha$ in crossover region

Using magnitude approximation we have magnitude for $\omega = 6$:

$\Longrightarrow \quad 50/\omega_c\alpha = 1 \Longrightarrow 50/6\alpha = 1 \Longrightarrow \alpha = 25/3 = 8.3$

Lag Compensation - 4

(5) Pick ω_1

We want to keep lag away from the action, i.e., crossover but not far away as mid-frequency may degrade. We can do this in two ways:

(a) Pick ω_1 so that $\alpha\omega_1 \approx \omega_c/10$

(b) Pick ω_1 so that $\partial\omega_1/\partial\alpha = 0$ (Pick largest ω_1 that satisfies phase margin requirements)

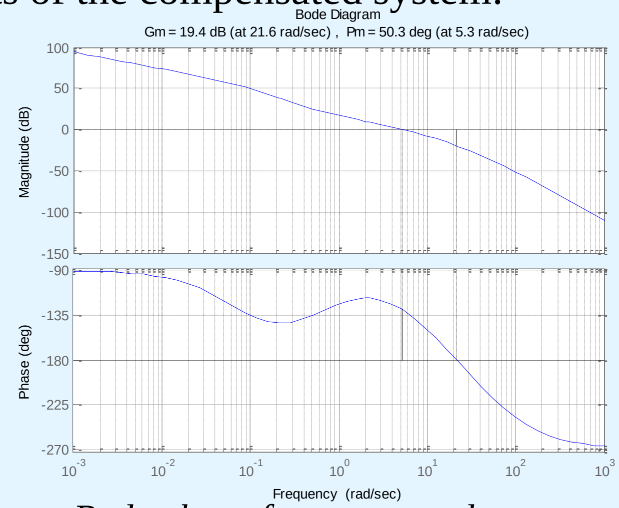
(a) Picking ω_1 using $\alpha\omega_1 = \omega_c/10$

So, if $\alpha\omega_1 = \omega_c/10 \Longrightarrow \omega_1 = \omega_c/10\alpha = 6/(10)(8.3) = .072$ r/s & $\alpha\omega_1 = (.072)(8.3) = .6$ r/s

Note: Setting $\alpha\omega_1 \approx (0.1\omega_c, 0.2\omega_c)$ will keep lag out of the way.

Therefore the compensator is: $H(s) = \dfrac{1 + s/.6}{1 + s/.072}$

Looking at the Bode plots of the compensated system:



Bode plots of compensated system

Lag Compensation - 5

Should also do step response, root locus and sensitivity analysis.

(b) Pick ω_1 so that $\partial\omega_1 / \partial\alpha = 0$

Note that having a bigger ω_1 & $\alpha\omega_1$ will mean a lower crossover frequency to makeup for phase lag introduced by $H(j\omega)$

We have three variables: α, ω_1 & ω_c and two equations.

1. $|KGH|_{\omega=\omega_c} = 1$ or $20\log_{10}|KGH|_{\omega=\omega_c} = 0$ db

$$\left|KG(j\omega_c)\right| = 1 \sim \frac{50(\omega_c/\alpha\omega_1)}{\omega_c(\omega_c/\omega_1)} = \frac{50}{\omega_c\alpha} \Rightarrow \frac{50}{\omega_c\alpha} = 1 \Rightarrow \omega_c = \frac{50}{\alpha}$$

2. $\measuredangle GH|_{\omega=\omega_c} = -\pi + \phi_m =$ given

$$-\pi/2 - \omega_c/10 - \omega_c/50 + [\pi/2 - \alpha\omega_1/\omega_c - \pi/2 + \omega_1/\omega_c] = -3\pi/4$$

$$-\pi/2 - 3\omega_c/25 - \omega_1 \cdot (\alpha - 1)/\omega_c = -3\pi/4$$

$$\Longrightarrow \pi/4 = 3\omega_c/25 + \omega_1/\omega_c \cdot (\alpha - 1) \Longrightarrow \omega_1 = [\pi/4 - 3\omega_c/25] \cdot \omega_c/(\alpha - 1)$$

Since, $\omega_c = 50/\alpha$, we have

$$\Longrightarrow \pi/4 = 6/\alpha + \alpha\omega_1/50 \cdot (\alpha - 1) \Longrightarrow \omega_1 = \frac{(\pi/4 - 6/\alpha)\,50}{\alpha(\alpha - 1)}$$

Let us now pick α to maximize ω_1 and, correspondingly, maximize the mid frequency attenuation caused by the lag compensation.

Since ω_1 is a function of α, we have

$$\frac{\partial\omega_1}{\partial\alpha} = 0 \Rightarrow \frac{6}{\alpha^2}\frac{50}{\alpha(\alpha-1)} - \frac{(2\alpha-1)(\pi/4 - 6/\alpha)50}{\alpha^2(\alpha-1)^2} = 0$$

Lag Compensation - 6

$$\frac{300}{\alpha} - \frac{(2\alpha-1)(\pi/4 - 6/\alpha)\,50}{(\alpha-1)} = 0$$

$$\approx 300/\alpha - 25\pi + 600/\alpha = 0$$

$$\Longrightarrow \frac{900}{25\pi} \approx 12$$

$\Longrightarrow$

$\alpha = 12$

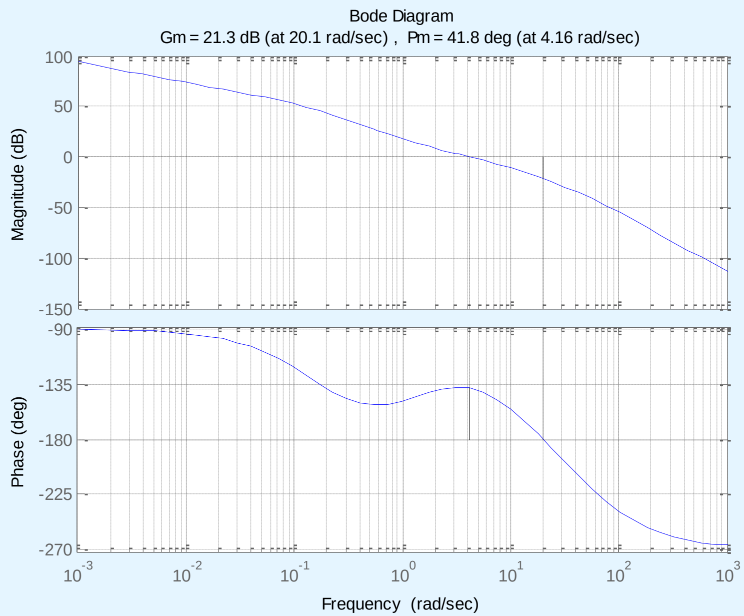
$\omega_c = 4.16$ rads/sec

$\omega_1 = .146$ rads./sec

$\alpha\omega_1 = 1.67$ rads/sec

MUST EVALUATE ACTUAL ω_c and ϕ_m

Therefore, the compensator H(s) is $H(s) = \dfrac{1 + s/1.752}{1 + s/.146}$



Bode plots of lag compensated sytem

Lag Compensation - 7

The simulated step responses for the compensated systems are shown in Fig III.23. However, should also draw root locus and look at sensitivity with respect to change in parameters, [K, α, ω_1, G(s), etc.]

<u>Some Comments:</u>

Suppose we were using gain compensation, K. The frequency ω^* for which
$\measuredangle KG(j\omega) = -135^o$ is found from:
$\measuredangle KG(j\omega^*) \sim -\pi/2 - \omega^*/50 = -3\pi/4 \implies \omega^* \approx 6.5$ rads/sec and at that frequency,
$|KG(j6.5)| \sim 50/6.5 \approx 7.6$

<u>Result:</u>

If pure gain compensation were to be used, the gain would be decreased by 7.6, which establishes a lower limit on the value of α to be used for the lag compensator. Also, ω^* establishes our upper limit on the crossover frequency ω_c for the lag compensator.

Lag Compensation - 8

Step responses of uncompensated system and lag compensated system

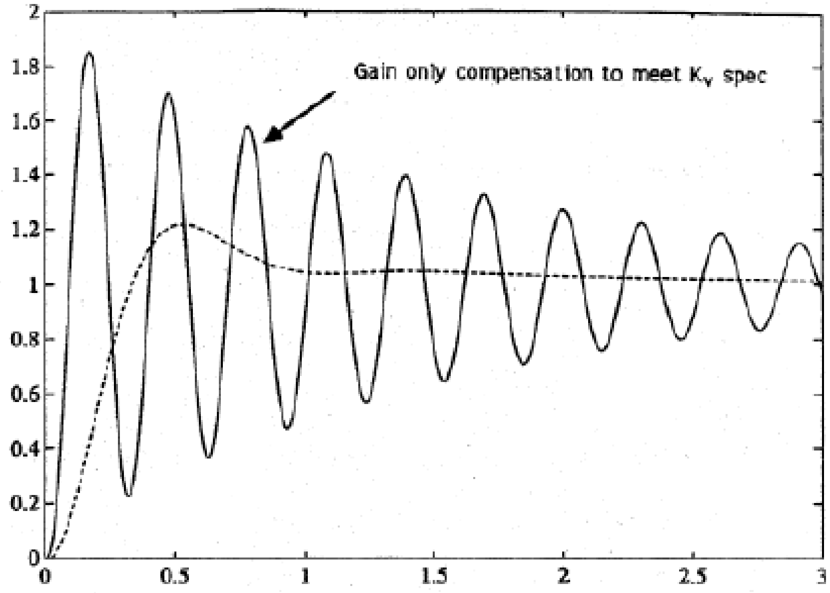


FIGURE III.23(a) STEP RESPONSE USING LAG COMPENSATOR (a)

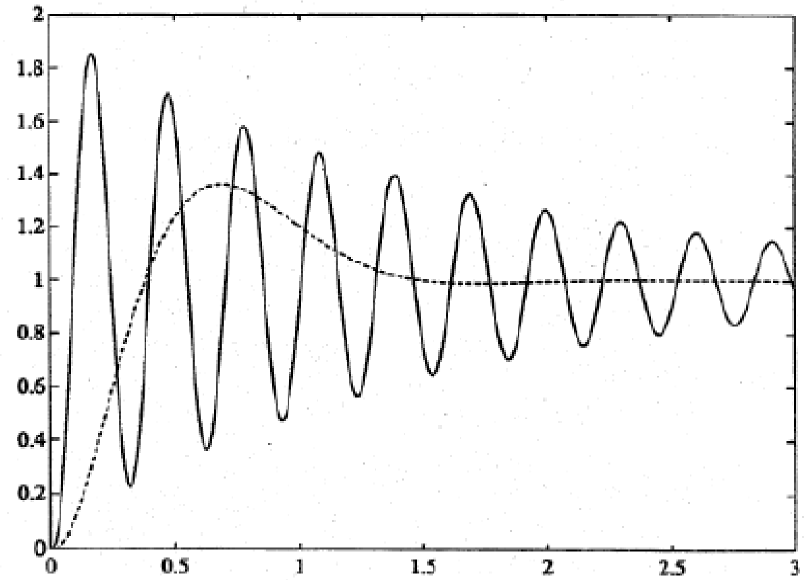


FIGURE III.23(b) STEP RESPONSE USING LAG COMPENSATOR

Review of Lag Compensation

Best way to get into it is to look at pros and cons of lag compensation.

Review of lag compensation

- Determine K for suitable K_v or Low-frequency range accuracy
- Sketch the Bode plot of KG(s)
- Decide need for compensation
- select appropriate region for crossover where $\angle KG(s) = -180^0 + \varphi_m + 5^0 - 10^0$
- Find gain reduction α needed in crossover region

==> compute α, ω_1 and ω_c

Basic Equations:

(1) $|K G(j\omega_c) H(j\omega_c)| = 1$ solve for ω_c in terms of α ($|H(j\omega_c)| \approx 1/\alpha$ around ω_c)

(2) $\measuredangle G(j\omega_c) + \underbrace{\measuredangle H(j\omega_c)}_{-(\alpha-1)\,\omega_1/\omega_c} = -180^0 + \phi_m + 5^0 - 10^0$

$-(\alpha-1)\,\omega_1/\omega_c$ radians is generally small $\approx 6^o$ or so.

(3a) $\alpha\omega_1 = \omega_c/10$

(OR)

(3b) maximize ω_1 w.r.t. α ==> $\partial\omega_1/\partial\alpha = 0$

PROS and CONS of Lag Compensation:

PROS	CONS
Lowers HF gain to help eliminate noise	Makes system more sluggish (adds lag, reduces BW)
Keeps good low freq. asymptote	Needs low crossover freq
High K_v	May reduce Mid Freq. gain
Built of Passive RC elements	Must have ϕ_m OK near intended ω_c

Lead Compensation - 1

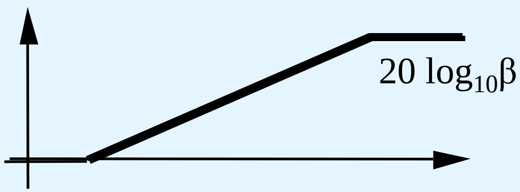
So lag network will not always work. Plant G(s) must have small $\measuredangle G(s)$ already.

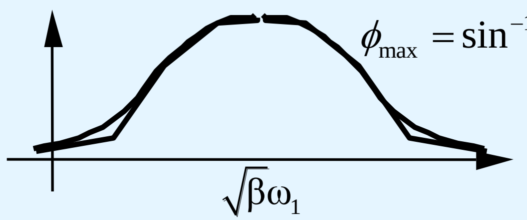
For example, if we have a system with $G(s)=1/s^2$, lag compensation is NO Good!!

Lag : Achieves ϕ_m by lowering ω_c via gain reduction. . . indirect approach

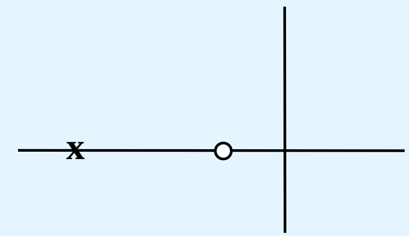
Lead: Adds phase lead directly in cross over region (direct approach)

Recall Lead: $H(s) = \dfrac{1 + s/\omega_1}{1 + s/\beta\omega_1}$; $\beta > 1$





$$\phi_{max} = \sin^{-1}\frac{\beta-1}{\beta+1}$$



As $\beta \to \infty$, $\phi_{max} \to 90^\circ$

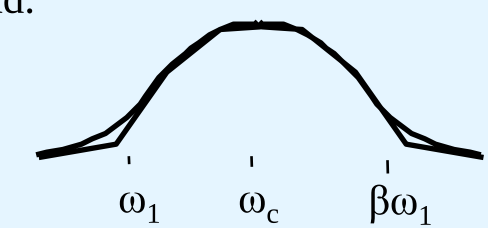
Note: adds HF gain β ==> increases gain crossover frequency ! !

==> increases BW of the system

Idea is to put ω_c at ω_{max} to get full benefit of phase lead.

For a given ϕ , $\beta = \dfrac{1 + \sin\phi}{1 - \sin\phi}$

Lead network straddles ω_c



Lead Compensation - 2

Example: $G(s) = \dfrac{10}{s\,(1 + s/10)}$

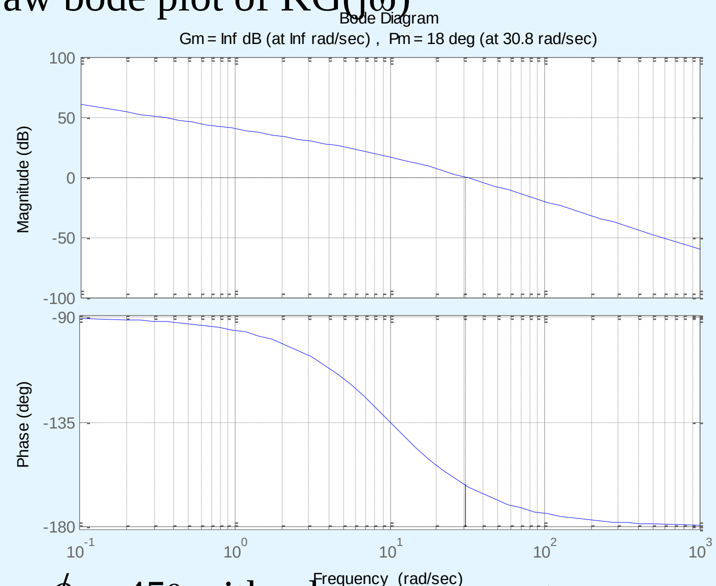
Specs:

1. $K_v = 100$
2. $|KGH| > 50$ for $\omega < 1$, i.e., 2% error for sine waves of frequency $\omega < 1$ rad/sec
3. $\phi_m \sim 45^o$

(1) Find K to meet LF requirements

$$K_v = \lim_{s\to 0} s\cdot KG(s) = \frac{10K}{(1)(1)} \geq 100 \quad ==> \quad K = 10$$

(2) Draw bode plot of $KG(j\omega)$



Bode plots of uncompensated sytem

To have $\phi_m = 45^o$ with a lag compensator, we could lower ω_c to about $\omega_c \leq 10$. however, this would violate mid-range requirements, since gain would be reduced by a factor of $1/\beta$.

Lead Compensation - 3

Therefore, use lead compensator to give required ϕ_m at present $\omega_c \approx 31.5$ r/s.

Recall lead compensator: $H(s) = \dfrac{1 + s/\omega_1}{1 + s/\beta\omega_1}$

(3) Find amount of additional phase needed

$\measuredangle KG = -162^o$ ==> $\phi_m = 18^o$ ==> need an additional 27º phase. But, since introduction of lead compensator will increase crossover frequency we will need more phase than initially anticipated. So, let us plan for increase in ω_c and hence more lead.

Pick $\phi_{max} = \pi/6 = 30^\circ$ So, required $\beta = \dfrac{1 + \sin 30^o}{1 - \sin 30^o} = 3$

Plan for an extra 5^0-10^0

(4) Pick ω_1

Place $\omega_c \approx \sqrt{\beta}\omega_1$. So, What to pick for ω_1?

Must be careful, since ω_1 & $\beta\omega_1$ are where the action is, unlike lag which is way out.

To find cross over $|KGH|_{\omega_c} = 1$ note: $|H|_{\omega_c} = \omega_c / \omega_1$ for $\omega_1 < \omega_c < \beta\omega_1$

$$\Rightarrow \frac{100}{\omega_c \dfrac{\omega_c}{10}} \cdot \frac{\omega_c}{\omega_1} = 1 \Rightarrow \frac{1000}{\omega_1 \cdot \omega_c} = 1 \Rightarrow \frac{1000}{\sqrt{\beta}\omega_1^2} = 1 \Rightarrow \omega_1 = \sqrt{1000/1.7} = 24.2 \text{ r/s}$$

$$\Rightarrow \sqrt{\beta}\omega_1 = \omega_c = (24.2)(1.7) = 41 \text{ r/s}$$

$$\beta\omega_1 = (24.2)(3) = 72.6 \text{ r/s}$$

Equivalently, find ω_c where

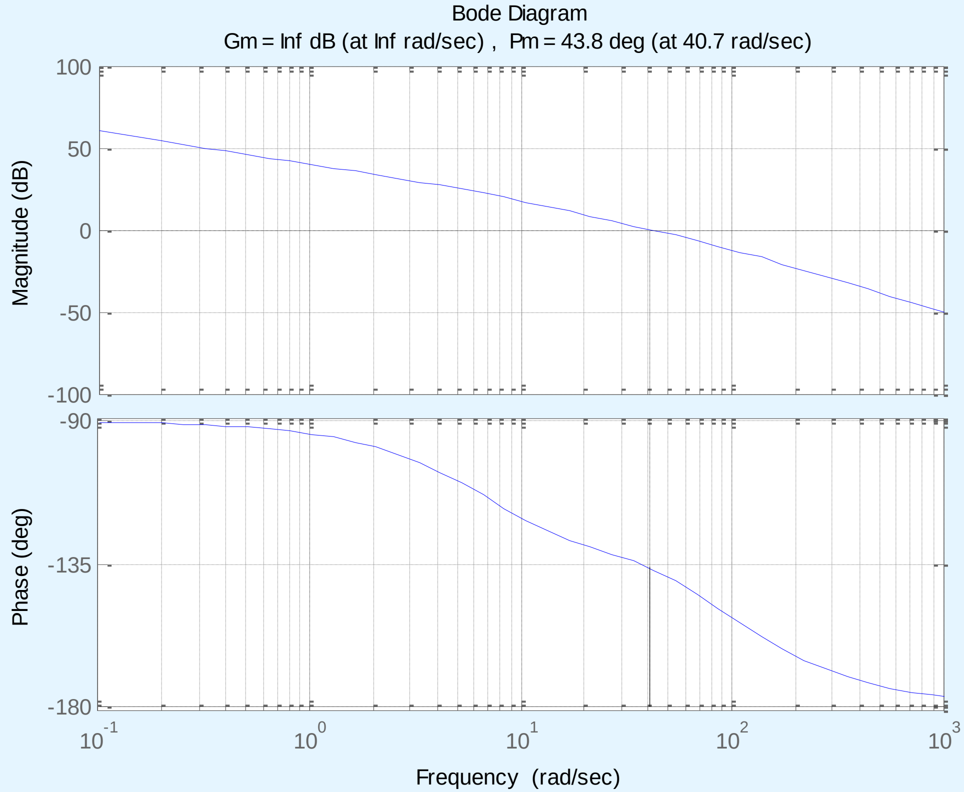
$$|KG(j\omega_c)| = \frac{1}{\sqrt{\beta}} = \frac{\omega_1}{\omega_c}$$

Lead Compensation - 4

Substituting the values of β and ω_1, the compensator H(s) is:

$$H(s) = \frac{1 + s/24.2}{1 + s/72.6}$$

Look at the Bode plots of the compensated system:



Bode plots of lead compensated sytem

Lead Compensation - 5

Note:

There is an increase in ω_c over the original crossover of 31.5 rad/sec. Recheck ϕ_m at new crossover frequency, i.e., did we plan ahead OK?

$\measuredangle KGH = -\pi/2 - \tan^{-1}(41/10) + \tan^{-1}(41/24) - \tan^{-1}(41/72) \sim -136^o$. So, $\phi_m = 44^o$

Shown below are the simulated step responses for the compensated and uncompensated system. Should also look at the root locus as well.

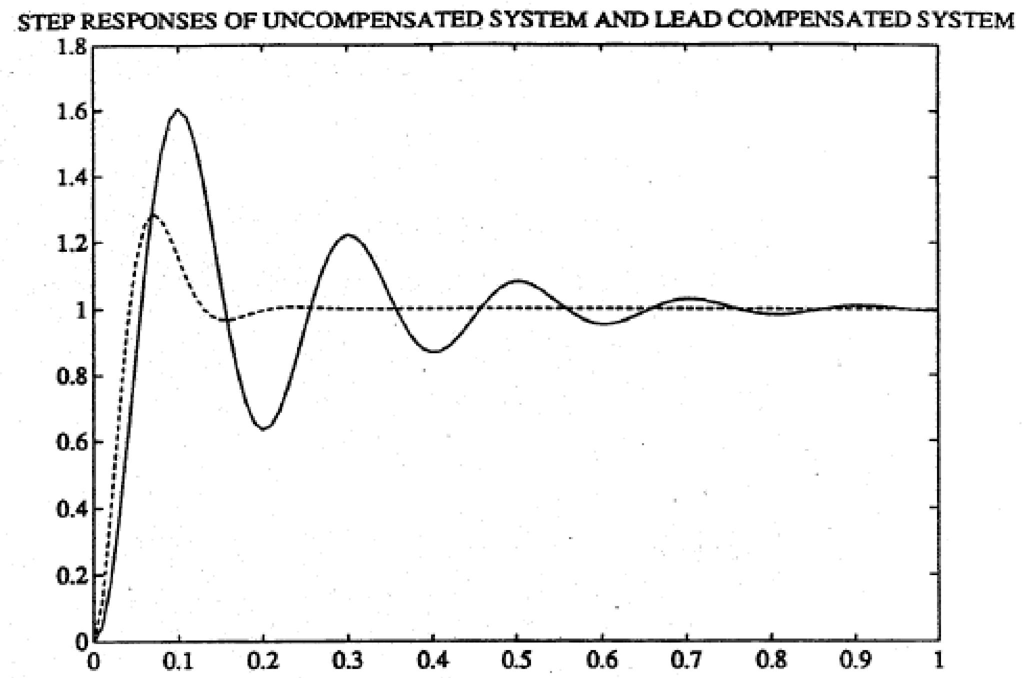


FIGURE III.26 SIMULATED STEP RESPONSE USING LEAD COMPENSATOR

Review of Lead Compensation

1. Determine K for K_v or mid-range gain
2. Sketch Bode of KG(s) & decide compensation
3. Pick ϕ_m required (plan for increased ω_c)
4. Equations $|KGH| = 1$ & $\omega_c = \sqrt{\beta}\omega_1$

use $$|H|_{\omega=\omega_c} = \frac{\omega_c}{\omega_1} = \frac{1}{|KG|_{\omega=\omega_c}} \quad \& \quad \beta = \frac{1+\sin\phi}{1-\sin\phi}$$

<u>PROS</u>

Higher BW & faster response

<u>CONS</u>

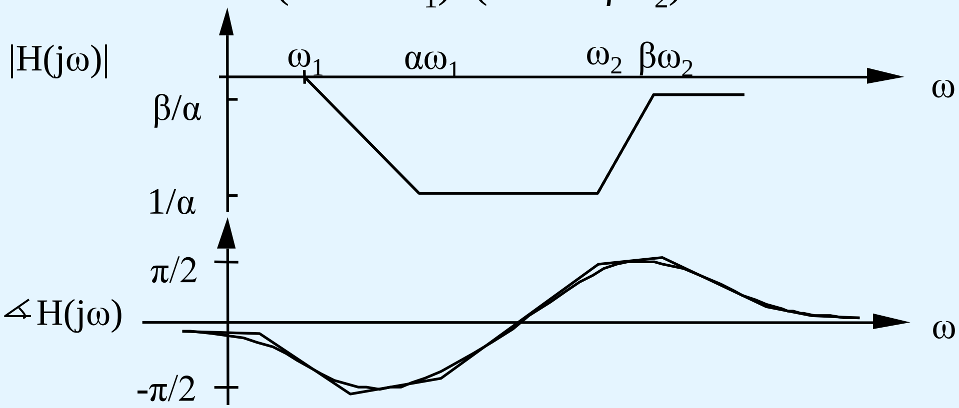
- Added gain increases crossover freq.
  $\Rightarrow$ we need more added phase than anticipated.
- Can't use lead if decrease in ϕ_m due to higher crossover > amount increased by lead network
- Usually don't like ω_c too big (noise BW)

Need system where $\measuredangle$ KG does not go down rapidly beyond crossover

Lag/Lead Compensation - 1

Recall that lead compensation increases the crossover frequency while lag compensation decreases it. Note that the advantages of a lag tend to be the disadvantages of a lead. So, good to use together as they are complementary.

$$H(s) = \frac{(1 + s/\alpha\omega_1)\ (1 + s/\omega_2)}{(1 + s/\omega_1)\ (1 + s/\beta\omega_2)} \qquad \alpha,\ \beta > 1 \quad ;\ \alpha\omega_1 < \omega_2$$



Bode plot for a lag-lead compensator

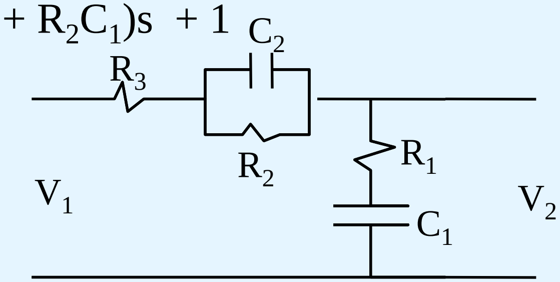
If $\beta/\alpha < 1$, then $|H(j\omega)| < 1\ \forall\ \omega$, H(s) can be realized by a passive RC network

$$\frac{V_2(s)}{V_1(s)} = \frac{(1 + R_1C_1s)(1 + R_2C_2s)}{(R_1C_1R_2C_2 + R_3C_1R_2C_2)\, s^2 + (R_1C_1 + R_2C_2 + R_3C_1 + R_2C_1)s + 1}$$

Comparing the last equations we have:

$$\alpha\omega_1 = \frac{1}{R_1C_1},\quad \omega_2 = \frac{1}{R_2C_2},\quad \frac{\beta}{\alpha} = \frac{1}{1 + R_3/R_1}\ \text{etc}$$

Often see $\alpha = \beta \implies R_3 = 0$



Lag/Lead Network

Lag/Lead Compensation - 2

Note:
- Get added phase lead from lead part (keep crossover between ω_2 and $\beta\omega_2$)
- keep crossover low from lag part since don't really offset high frequency gain by much (β/α)
- Design technique is almost total trial & error to pick ω_1 , ω_c , α, β, ω_2 etc.

Design approach:

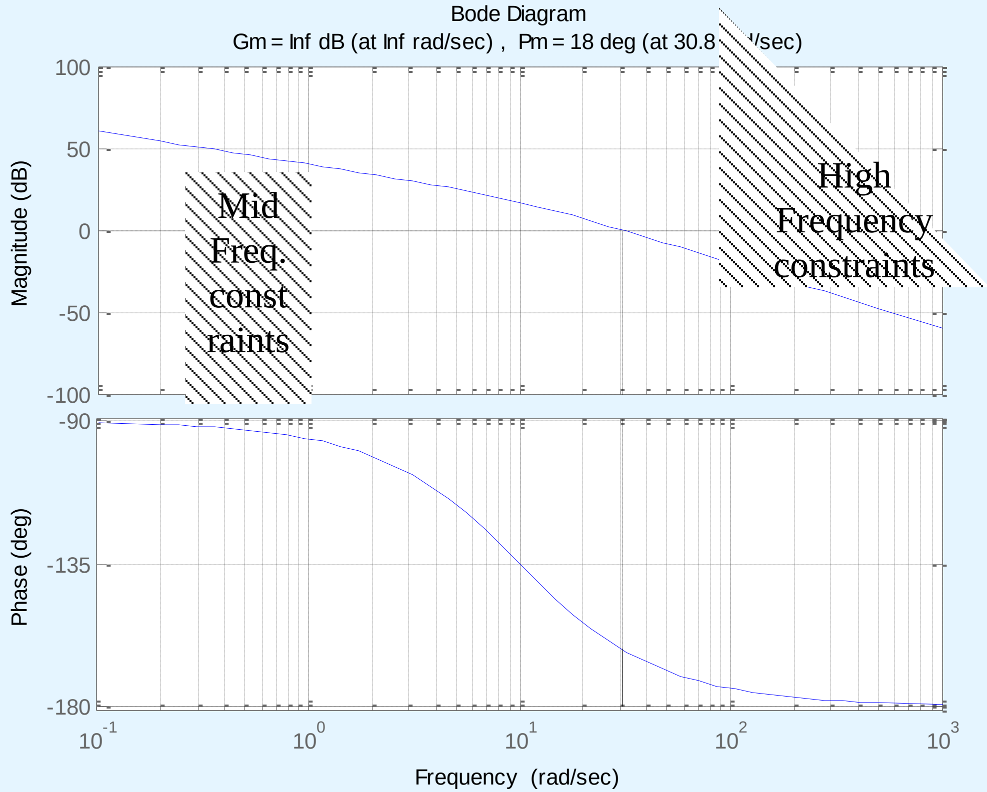
1. Pick K for LF accuracy & plot K G(jω)
2. Locate approximate ω_1 via mid frequency requirements
3. Locate approximate $\beta\omega_2$ via HF requirements
4. Put in $\alpha\omega_1$, ω_2 to locate crossover frequency to get good ϕ_m & "fair stretch of –20 db slope" near crossover.

Example: $$G(s) = \frac{10}{s(1 + s/10)}$$

Specifications:
1. $K_v = 100$
2. $\phi_m \sim 45^o$
3. < 2 % error for sinusoidal inputs up to ω = 1 rads/sec.
4. sinusoidal inputs of greater than 100 rads/sec should be attenuated to less than 5% at the output.

Design:
- spec 3 => $|KGH| > 50$ for $\omega < 1$, or $20\log_{10}|KGH| > 34$ db for $\omega < 1$
- spec 4 => $\frac{|y|}{|r|} = \frac{|KGH|}{|1+KGH|} = .05$ for $\omega > 100$ rads/sec $\Rightarrow$ <-26dB
- K = 10 for correct K_v

Lag/Lead Compensation - 3



K = 10

$$KG(s) = \frac{100}{s(1 + s/10)}$$

I have found the following iterative process to work well for lead-lag design.

Step 1: Select a cross-over frequency, ω_c

Step 2: Do the lead design to satisfy phase margin requirements

Step 3: Do the lag design to maintain phase margin

Step 4: Go to step 1 and repeat with a different ω_c until the mid-frequency and high-frequency constraints are met

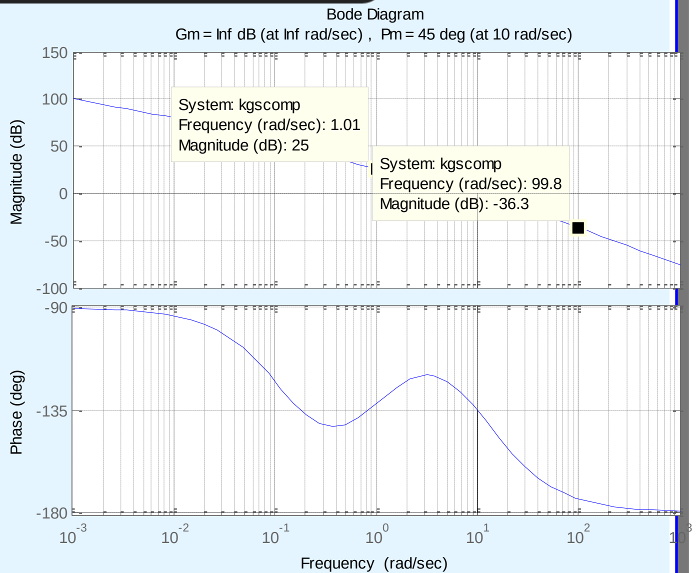
Lag/Lead Compensation - 4

- Design 1: Let ω_c = 10 rad/'sec

$$Lead: H_{Lead}(s) = \frac{0.1091\ s+1}{0.09168\ s+1}; Lag:\ \ H_{Lag}(s) = \frac{s+1}{7.737\ s+1}$$

$$H(s) = H_{Lag}(s)H_{Lead}(s);\ \varphi_m = 45^0$$

Fail: 25dB @ 1 rad/sec; -36.3dB @ 100 rad/sec. Need higher ω_c

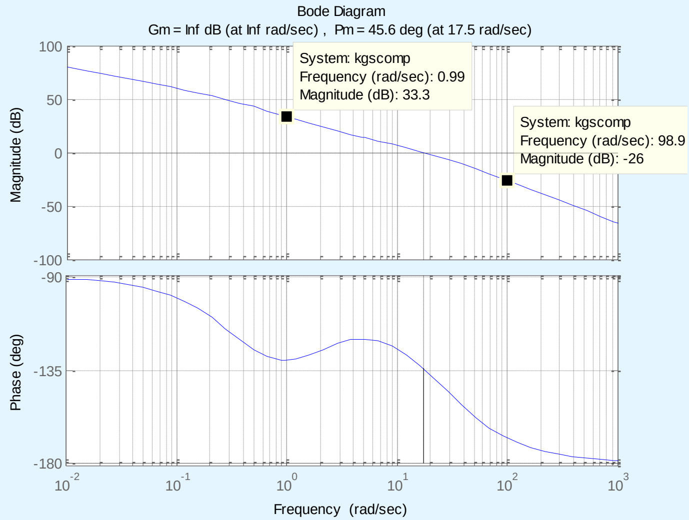


- Design 2: Let ω_c = 17.5 rad/sec

$$Lead: H_{Lead}(s) = \frac{0.08194\ s\ +\ 1}{0.03985\ s\ +\ 1}; Lag:\ \ H_{Lag}(s) = \frac{0.5714\ s\ +\ 1}{2.331\ s\ +\ 1}$$

$$H(s) = H_{Lag}(s)H_{Lead}(s);\ \phi_m = 45.6^0$$

Pretty close: 33.2dB @ 1 rad/sec; -26.6dB @ 100 rad/sec.



Numerical Optimization (e.g., via genetic algorithm):

$$\min_{K,\alpha,\beta,\omega_1,\omega_2} \int_0^\infty |t^\alpha e(t)|^p\ dt; \alpha=0, p=1 \Rightarrow IAE; \alpha=0, p=2 \Rightarrow ISE$$

$\alpha = 1, p = 1 \Rightarrow$ Integral absolute time-weighted error ($IATE$)

$\alpha = 1, p = 2 \Rightarrow$ Integral squared time-weighted error ($ISTE$)

subject to :

$(i)\ 180^0 + \angle G(j\omega_c)H(j\omega_c) \geq \phi_m$ $\qquad$ $(v)\ B^- \leq u(t) \leq B^+$

$(ii) K \lim_{s\to 0} sG(s) \geq K_v$

$$(iii) K|G(j\omega)\frac{(1+j\omega/\alpha\omega_1)}{(1+j\omega/\omega_1)}\frac{(1+j\omega/\omega_2)}{(1+j\omega/\beta\omega_2)}| \geq G_{mf}\ \ \forall\omega \leq \omega_{mf}$$

$$(iv) K|G(j\omega)\frac{(1+j\omega/\alpha\omega_1)}{(1+j\omega/\omega_1)}\frac{(1+j\omega/\omega_2)}{(1+j\omega/\beta\omega_2)}| \leq G_{hf}\ \ \forall\omega \geq \omega_{hf}$$

Additional Examples - 1

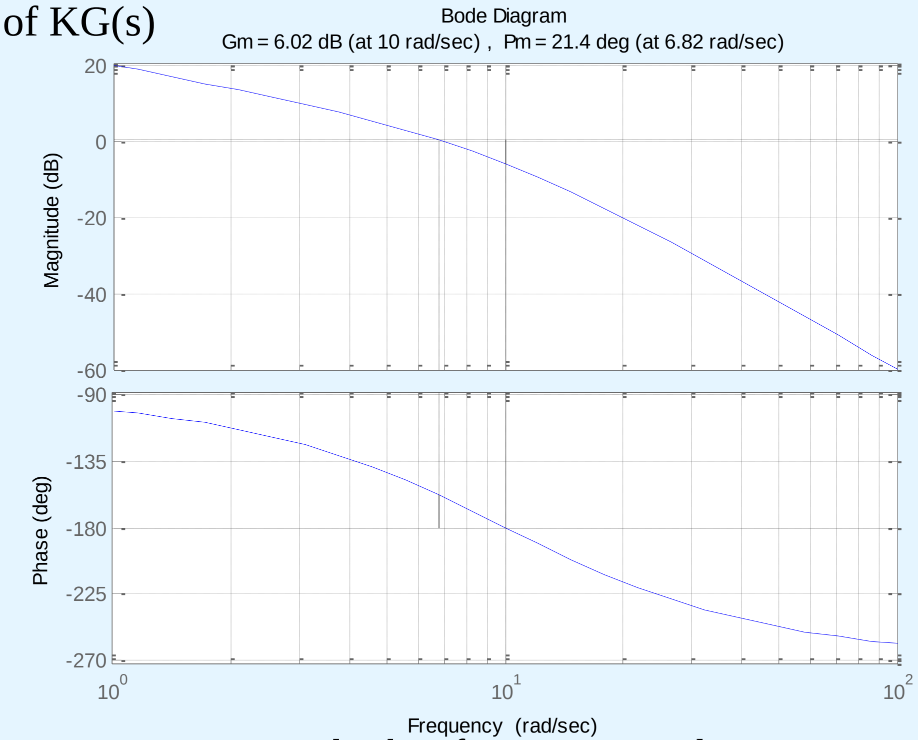
EXAMPLE 1

Consider a unity feedback system with the following plant transfer function: $G(s) = \dfrac{1}{s(1 + s/10)^2}$

Design a suitable compensator such that the overall control system meets the following specs: (1) $K_v \geq 10$, (2) $\phi_m \geq 45^o$.

(1) $K_v = \lim_{s \to 0} sKG(s) \geq 10 \Rightarrow K = 10$

(2) Bode plots of KG(s)



Bode plot of uncompensated system

Additional Examples - 2

(3) The phase margin requirement is not met. By using a lag compensator, we can lower the crossover frequency to obtain the desired phase margin.

(4) Determine ω_c

$$-\pi/2 - \omega_c/10 - \omega_c/10 = -3\pi/4$$

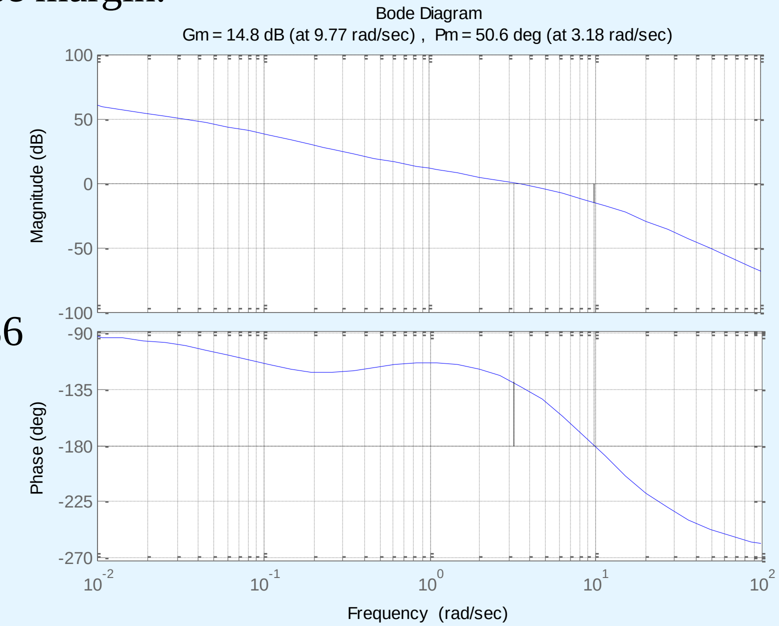
$$\omega_c/5 = \pi/4 \implies \omega_c = 5\pi/4 = 3.93 \text{ r/s}$$

(5) Pick α

$$|KG(j\omega)H(j\omega)|\Big|_{\omega_c = 3.93} = 1 \quad \frac{10}{\omega_c \alpha} = 1 \implies \alpha = 2.86$$

(6) Pick ω_1

$$\alpha\omega_1 = \omega_c/10 \implies \omega_1 = .122$$



Therefore the compensator is

$$H(s) = \frac{(1 + s/.35)}{(1 + s/.122)}$$

With this compensator the crossover frequency is 3.2 r/s giving a phase margin of 50°. Shown in figure is the root locus of the closed loop system. Also shown in figure is the simulated step response of the system. Should also examine sensitivity.

Additional Examples - 3

- Root locus of closed-loop system

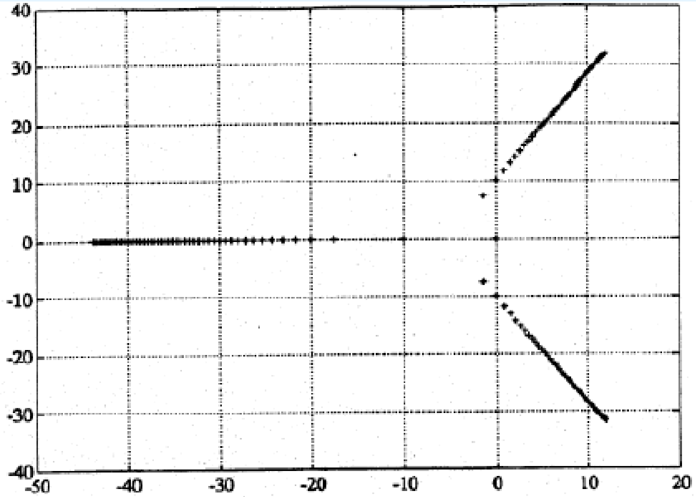


FIGURE III.31 (a) ROOT LOCUS OF UNCOMPENSATED SYSTEM

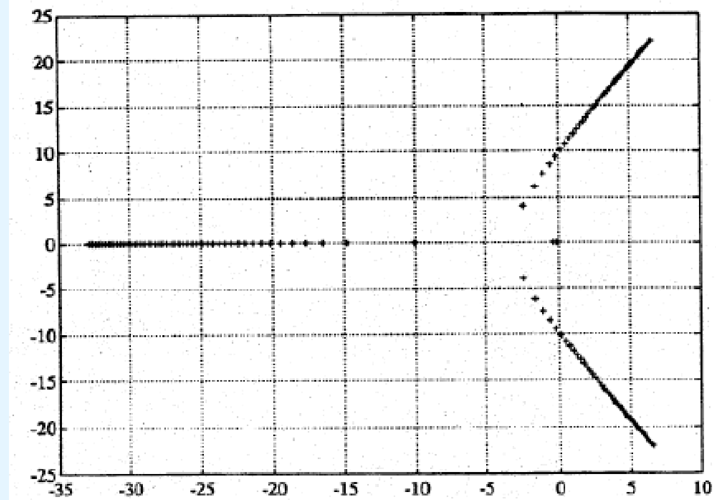


FIGURE III.31 (b) ROOT LOCUS OF LAG COMPENSATED SYTEM

- Simulated step responses of closed-loop system

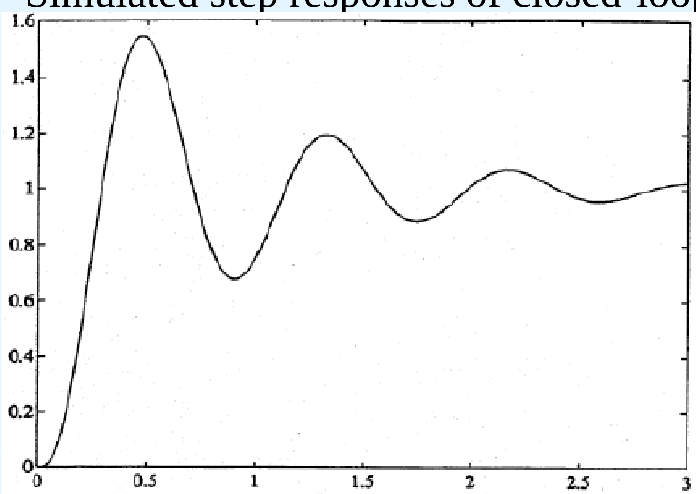


FIGURE III.32 (a) UNCOMPENSATED SYSTEM RESPONSE

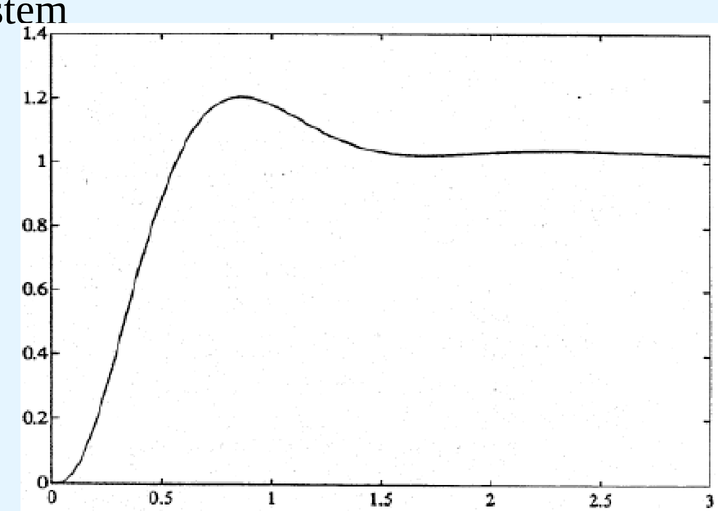


FIGURE III.32 (B) LAG COMPENSATED SYSTEM RESPONSE

Additional Examples - 4

EXAMPLE 2

Design a suitable compensator which meets the specs for the following system:

$$G(s) = \frac{K}{s(s+5)(s+10)}$$

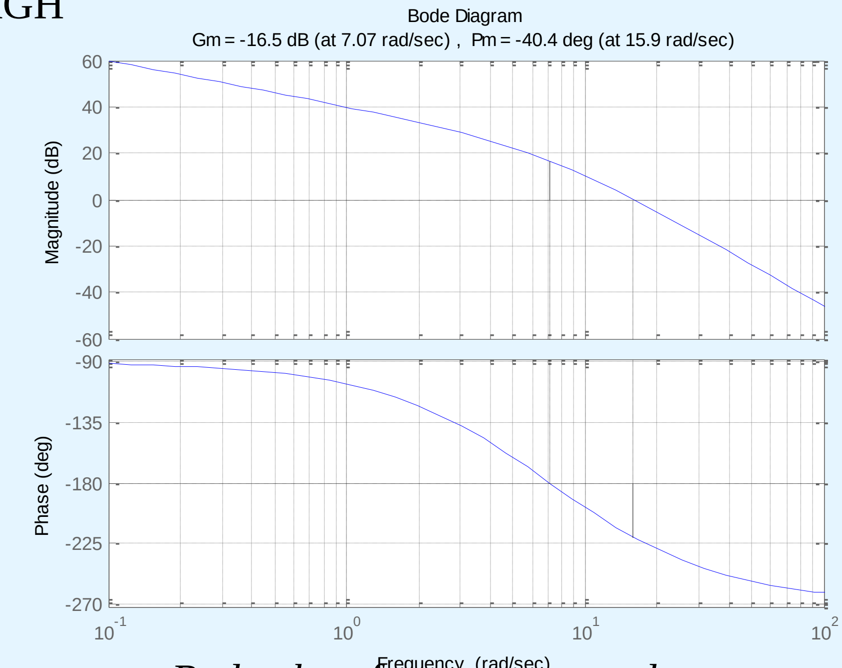
(1) $K_v = 100$

(2) $\phi_m = 45^o$

(1) We must make K = 5000 in order to get $K_v = 100$.

$$KG(s) = \frac{5000}{s(s+5)(s+10)}$$

(2) Bode Plot KGH



Bode plot of uncompensated system

Additional Examples - 5

The phase margin specification is not satisfied. Moreover, it is not possible to design a lead compensator since the phase shift is very large for negative gain. A lag compensator does exist, but it requires a pole very near the origin, with a pole-zero ratio of more than thirty. In practice, a pole very close to origin is not desirable, since the corresponding compensator would require an RC network with a large time constant.

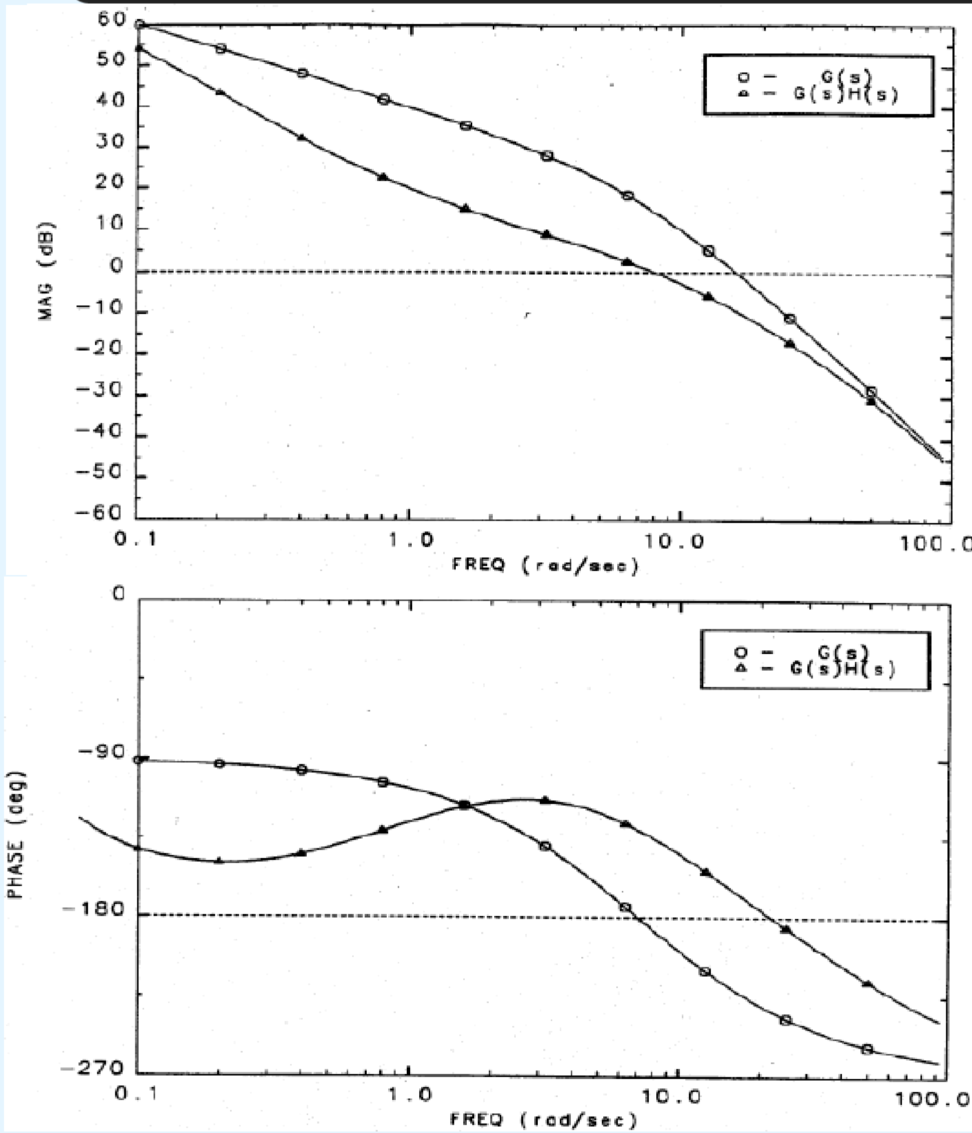
A lag-lead compensator can, however, be obtained by selecting a crossover frequency between 4 and 10. For $\omega_c = 8$, the following compensator is obtained, with $\alpha = \beta = 13.09$.

$$H(s) = \frac{(s+3.422)(s+.7587)}{(s+44.5)(s+.058)}$$

This compensator gives the desired phase margin of 45^o. The simulated step response are Shown in the figure.

Additional Examples - 6

- Bode plot



Additional Examples - 7

- Root locus of closed-loop system

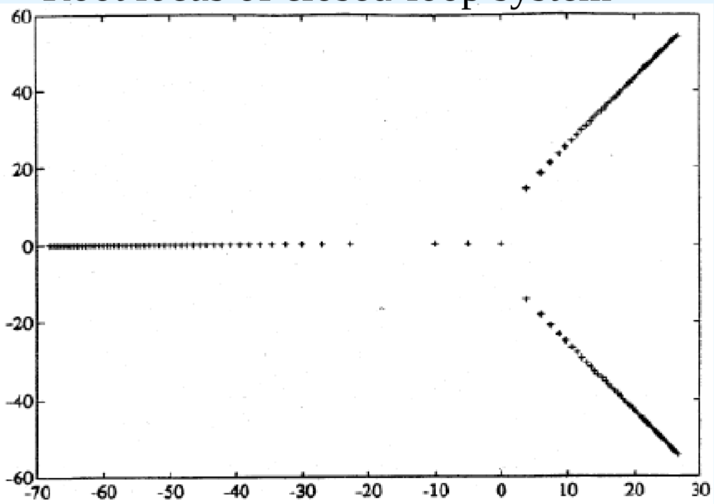


FIGURE III.34 (a) ROOT LOCUS OF UNCOMPENSATED SYSTEM

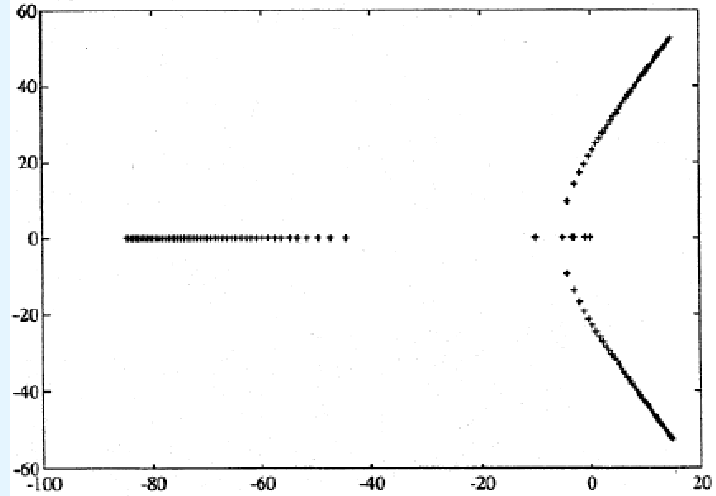


FIGURE III.34 (b) ROOT LOCUS OF LAG/LEAD COMPENSATED SYSTEM

- Simulated step responses of closed-loop system

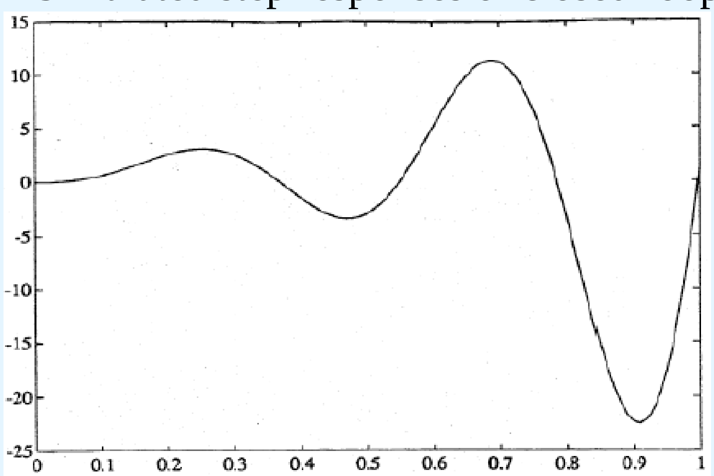


FIGURE III.35 (a) STEP RESPONSE OF UNCOMPENSATED SYTEM

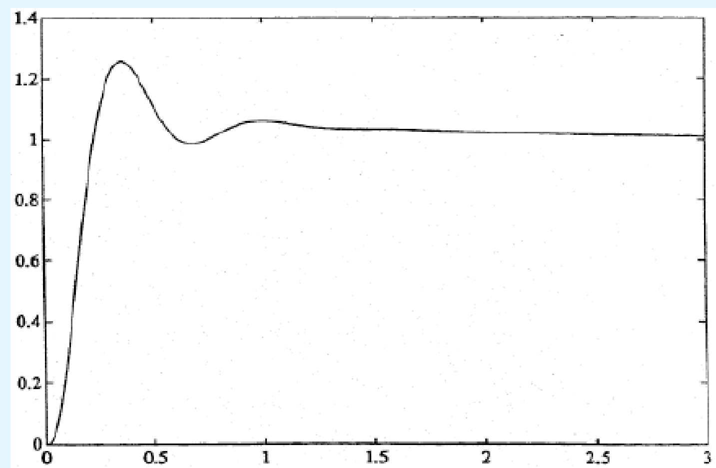


FIGURE III.35 (b) STEP RESPONSE OF LAG/LEAD COMPENSATED SYTEM

Additional Examples - 8

EXAMPLE 3

A unity feedback system has the following transfer function: $G(s) = \dfrac{1}{s^2}$

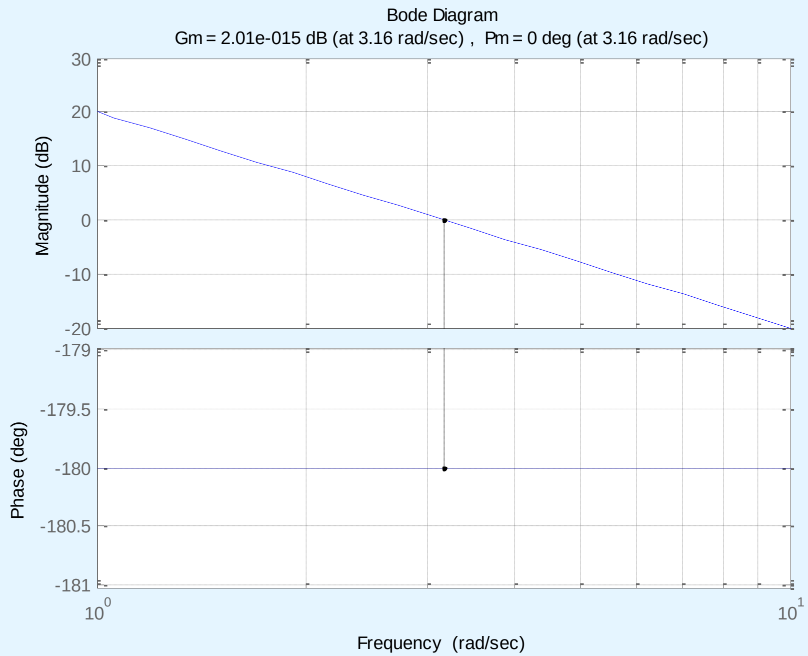
Find a compensator to meet the following specifications:

Specs: (1) $K_a \geq 10$

(2) $\phi_m \geq 45^o$

(1) $K_a = \lim_{s \to 0} s^2 KG(s) \geq 10 \implies K = 10$

(2) Bode of KG



Bode plot of uncompensated system

Additional Examples - 9

Since the phase margin is 0°, a lag compensator will not work. By using a lead compensator We can add 45° phase at the crossover frequency.

(3) Determine β

Anticipating a few degrees of phase due to the compensator:

[Note this is not necessary since $\measuredangle G((j\omega)$ is flat everywhere!]

$$\beta = \frac{1 + \sin 48^o}{1 - \sin 48^o} \quad ==> \quad \beta = 6.786$$

(4) |KGH| = 1 using |H| = ω_c / ω_1 and $\omega_c = \sqrt{\beta}\, \omega_1$

$$\frac{10}{(\omega_c)^2} \cdot \frac{\omega_c}{\omega_1} = 1 \quad ==> \quad \omega_1^2 = 10/\sqrt{\beta}$$

$$==> \quad \omega_1 = 1.959$$

$$==> \quad \beta\omega_1 = 13.29$$

Therefore the compensator is:

$$H(s) = \frac{(1 + s/1.959)}{(1 + s/13.29)}$$

With this compensator the crossover frequency is 5.1 r/s resulting in a phase margin of 48°. The root locus of the closed loop system is shown in figure and the step response is shown in the next figure.

Additional Examples - 10

- Root locus of closed-loop system

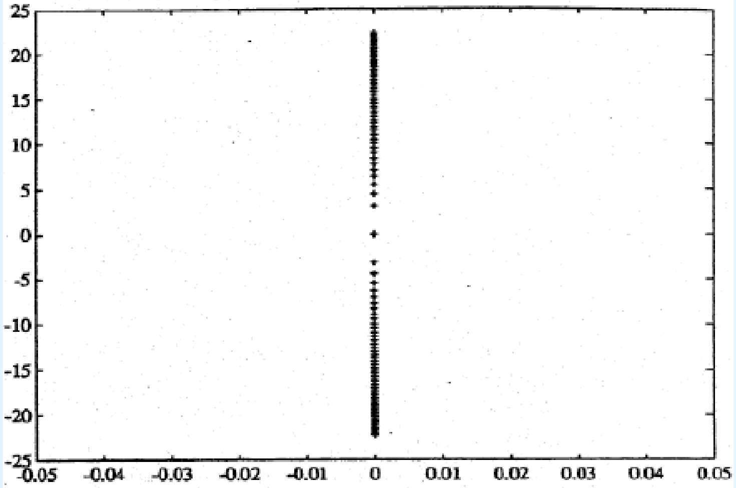


FIGURE III.37 (A) ROOT LOCUS OF UNCOMPENSATED SYTEM

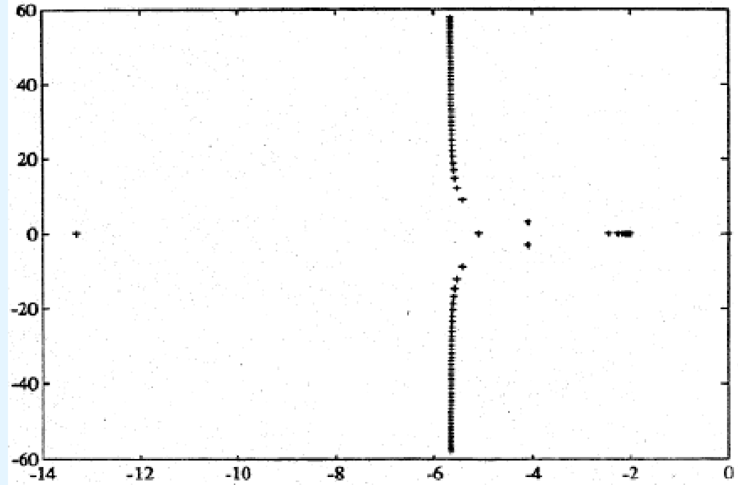


FIGURE III.37 (B) ROOT LOCUS OF LEAD SYTEM

- Step responses of uncompensated system and lead compensated system

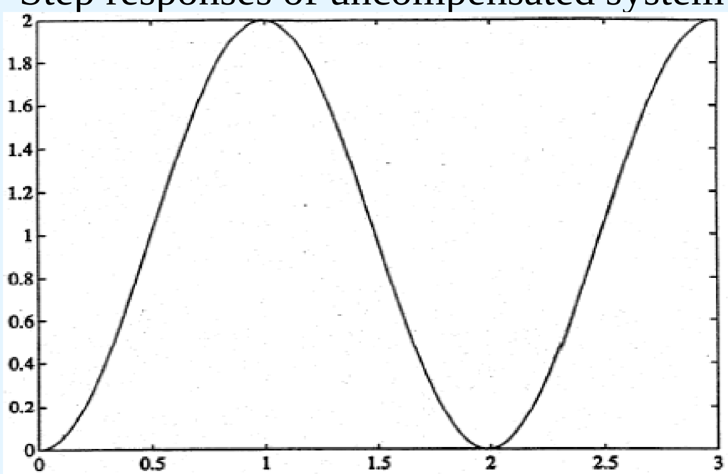


FIGURE III.38 (a) STEP RESPONSE OF UNCOMPENSATED SYSTEM

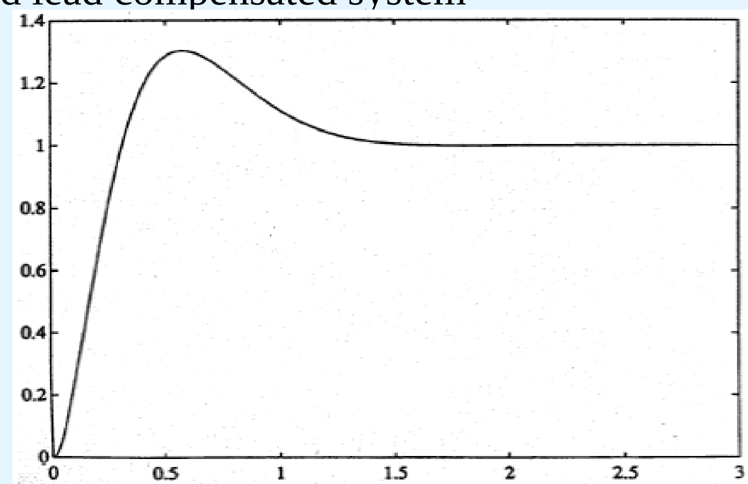


FIGURE III.38 (b) STEP RESPONSE OF LEAD COMPENSATED SYSTEM

Critique of Bode-based H(s) Designs

- Classical design techniques are simple to use.
  - graphical techniques
  - some trial and error
- Designs are easy to implement via analog circuitry.
- Consider Lag-lead compensator when neither alone will suffice.

  $\Rightarrow$ pick ω_2, β, ω_1, α
- Most-used design technique
  - there are many such compensators "out there"
  - can they be modified for digital implementation?

$$H(s) \rightarrow \tilde{H}(z)$$

But there are limitations -

- Simple lag, lead, etc., may not be sufficient.
- High-order compensator design via Bode, or root locus, is a challenging process, especially for humans.
- Compensation does not use all available info
  - uses only $y(t)$, not states $\underline{x}(t)$
- Difficult to extend procedure to multi-input, multi-output systems.

(Personally, I prefer Bode design approach over root locus.)

PID (Proportional-Integral-Derivative) Controller

- Most common packaged form of controller
  - Very popular in process control industry
- Continuous PID, $u(s) = H(s)e(s)$,

$$u(s) = K\left[1 + \frac{1}{T_1 s} + \frac{T_2 s}{1 + T_2 s/N}\right] e(s)$$

Proportional Term — Integral Term — Derivative Term

T_1 = integral or reset time (big number usually)
T_2 = derivative time
$N \approx 2 \rightarrow 20$ (usually fixed) (derivative gain)

- Integral term not necessary if there is an integrator (k/s) in the loop already.
- Equivalent to lead compensator (PD part) + integral term

$$\text{PD:} \quad 1 + \frac{T_2 s}{1 + T_2 s/N} \Leftrightarrow \frac{1 + s/\omega_2}{1 + s/\beta\omega_2}$$

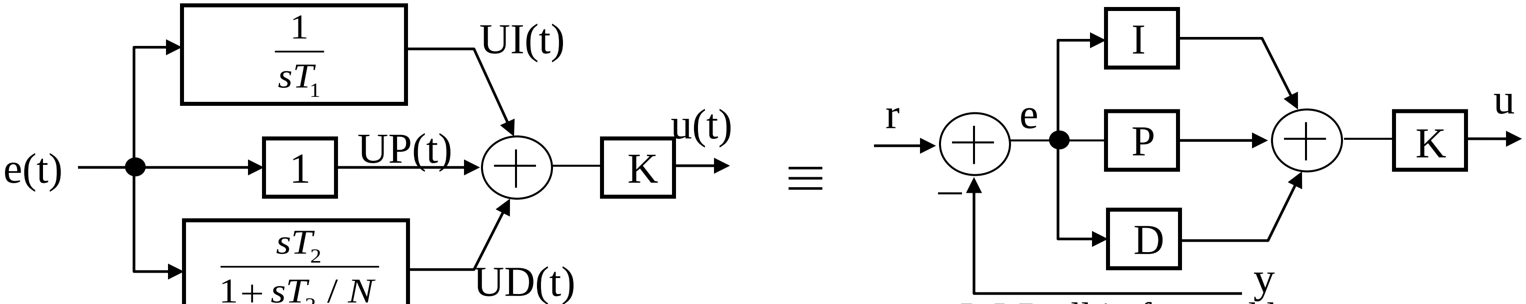
$$\text{with} \quad \left.\begin{matrix} N = \beta - 1 \\ T_2 = \dfrac{\beta - 1}{\beta\omega_2} \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} \beta = N + 1 \\ \omega_2 = \dfrac{N}{(N+1)T_2} \end{matrix}\right.$$

$$\beta\omega_2 = \frac{N}{T_2}$$
$$\omega_2 = \frac{N}{(N+1)T_2}$$
$$\beta = N + 1$$

- Various "tuning rules" for T_1, T_2, K exist.
  - Ziegler and Nichols (1942)
- Not all parts are necessary for good control (e.g., P, PI, PD, …)

PID Controller Configurations - 1

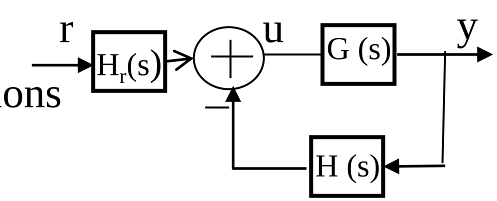
- Implementation – “Textbook”   Sum up 3 parts separately:



P-I-D all in forward loop

$$UI(t) = \frac{1}{T_1}\int_0^t e(\tau)d\tau; UP(t) = e(t); \frac{d}{dt}UD(t) = -\frac{N}{T_2}UD(t) + N\frac{d}{dt}e(t)$$

- Alternate Implementations for proportional and derivative actions



Two DOF P-I-D

$$UP(t) = br(t) - y(t); \frac{d}{dt}UD(t) = -\frac{N}{T_2}UD(t) + N\frac{d}{dt}[cr(t) - y(t)]$$

$b = set\ point\ weighting\ for\ proportional\ control\ (b<1) \Rightarrow extra\ zero\ for\ N \to \infty \Rightarrow \%\ overshoot\ goes\ down$

$c\ =\ set\ point\ weighting\ for\ derivative\ control\ (often\ set\ to\ zero)$

- with b and c, you have two degrees of freedom in a PID

$$u(s) = H_r(s)r(s) - H(s)y(s) \text{ where } H_r(s) = K[b + \frac{1}{T_1 s} + c\frac{T_2 s}{1+T_2 s/N}] \ \&$$

$$H(s) = K[1 + \frac{1}{T_1 s} + \frac{T_2 s}{1+T_2 s/N}] \Rightarrow T(s) = \frac{G(s)H_r(s)}{1+G(s)H(s)}$$

$$\Rightarrow H_r(s) \text{ gives extra zeros as roots of } T_1T_2(c+\frac{b}{N})s^2 + (T_1 b + \frac{T_2}{N})s + 1 = 0$$

$b = c = 0 \Rightarrow zero\ at\ -\frac{N}{T_2} \to -\infty\ as\ N \to \infty$

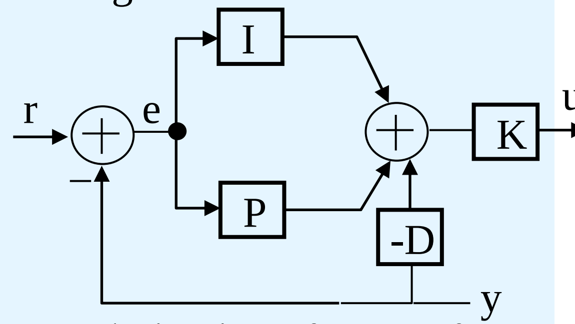
$\Rightarrow no\ extra\ zeros$

$c = 0 \Rightarrow zeros\ at\ -\frac{1}{bT_1}\ and\ -\frac{N}{T_2}$

$\Rightarrow one\ zero\ at\ -\frac{1}{bT_1}\ as\ N \to \infty$

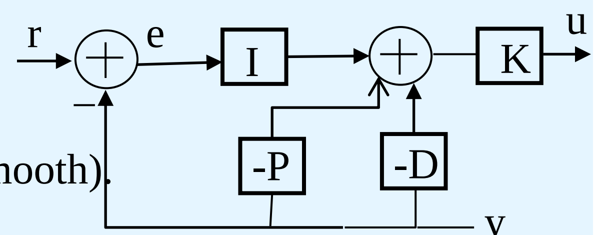
PID Controller Configurations - 2

- $c = 0 \Rightarrow$ **derivative of output form**
  - If r suddenly changes, e.g., a step change, then *de/dt* may be large and UD will have a "spike" at time t. This is undesirable.
  - So, modify UD computation to use only dy/dt.
  - Since y(t) cannot change too much, UD will be OK.
  - CL stability is unaffected (stability not a function of r).
  - Often times, y(t) is filtered via $G_f(s) = 1/(1 + sT_f)$



"derivative of output form"

- $b = c = 0 \Rightarrow$ "**set-point on I" structure**
  - Move P to act only on y also, UP = –y(k)
  - Only integral compensation uses error signal.
  - Popular in process control (keeps control signal very smooth).



"set-point on **I** structure"

- **Series form** $(N \to \infty)$

$$H(s) = K[1 + \frac{1}{T_1 s} + T_2 s] = K'(1 + \frac{1}{T_1' s})(1 + T_2' s)$$

$$\Rightarrow T_1 = T_1' + T_2'; K = K'\frac{T_1}{T_1'}; T_2 = \frac{T_1' . T_2'}{T_1}$$

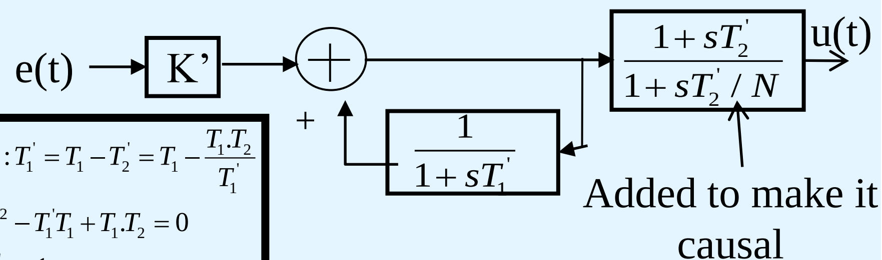
$$\Rightarrow K' = \frac{K}{2}\left(1 + \sqrt{1 - 4T_2/T_1}\right); T_1' = \frac{K' T_1}{K}; T_2' = \frac{T_1 T_2}{T_1'}$$

$Note: T_1 > T_1', K > K' \ while \ T_2 < T_2'$

$$Note: T_1' = T_1 - T_2' = T_1 - \frac{T_1 . T_2}{T_1'}$$

$$\Rightarrow T_1'^2 - T_1' T_1 + T_1 . T_2 = 0$$

$$\Rightarrow \frac{T_1'}{T_1} = \frac{1}{2}\left(1 + \sqrt{1 - 4T_2/T_1}\right)$$



"series form with automatic reset"

Integral Windup Modifications - 1

- A problem that arises when u is limited, e.g.,

$$B^- \leq u(t) \leq B^+$$

(symmetric limits are most common, $B^- = -B^+$)

- Limits are imposed by the system under control, e.g., actuator constraints.
  - Match these limits in controller software:

if ($u \geq B^+$) set $u = B^+$, flag = +1
if ($u \leq B^-$) set $u = B^-$, flag = –1
else flag = 0

- The control probably saturated because e(t) was large.
  - Because u is limited the error *e* will not be reduced to zero as fast (slower system).
  - This is not indicative of a steady-state *e*.

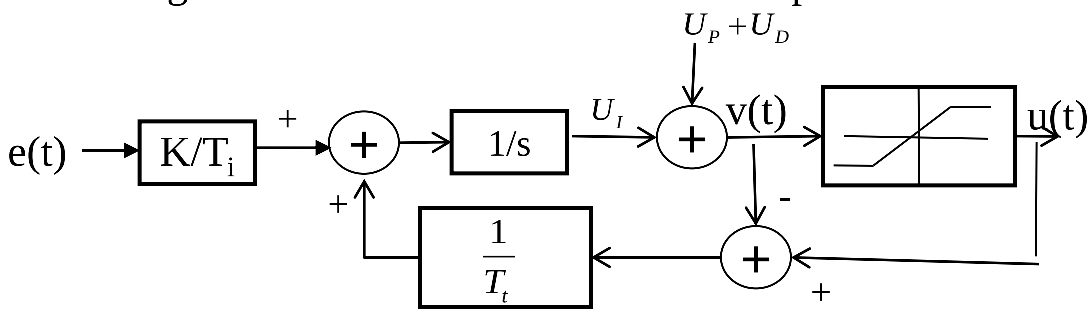
  => Turn off/skip the integration of e(t) in UI if the last control value was at a limit
  Conditional integration: if (flag = 0) do integration, else skip integration

- Integral protection
  - Value of UI does not change if/when u is saturated.

- Include PID structures in Cntrl subroutine, OPT = 4 (parallel),5 (derivative),6 (set point), 7(2 DOF),…..

Integral Windup Modifications - 2

- Tracking or back calculation to avoid windup



v(t) = u(t) ⇒ no feedback
v>u ⇒ makes UI less positive
v<u ⇒ makes UI less negative
$T_d < T_t < T_i$

- Further integration term modifications
  - UI removes ss error, but introduces –90° phase lag => T_1 ~ large.
  - Common to limit |UI|, e.g., |UI| < M.
  - Consider integrating only when e is small (pros & cons)
- Alternate implementation forms
  - "velocity" form: computes Δu. Best implemented digitally (see Lectures 9 and 10)
  - "bumpless" transfer: for changing manual ↔ auto mode. This is accomplished via "velocity" form and tracking form
- Systems with delays
  - Couple P-I(-D) with a Smith predictor
- Systems with oscillatory and unstable poles
  - If you have to use PID, use set point on I structure. Need more complicated controllers.

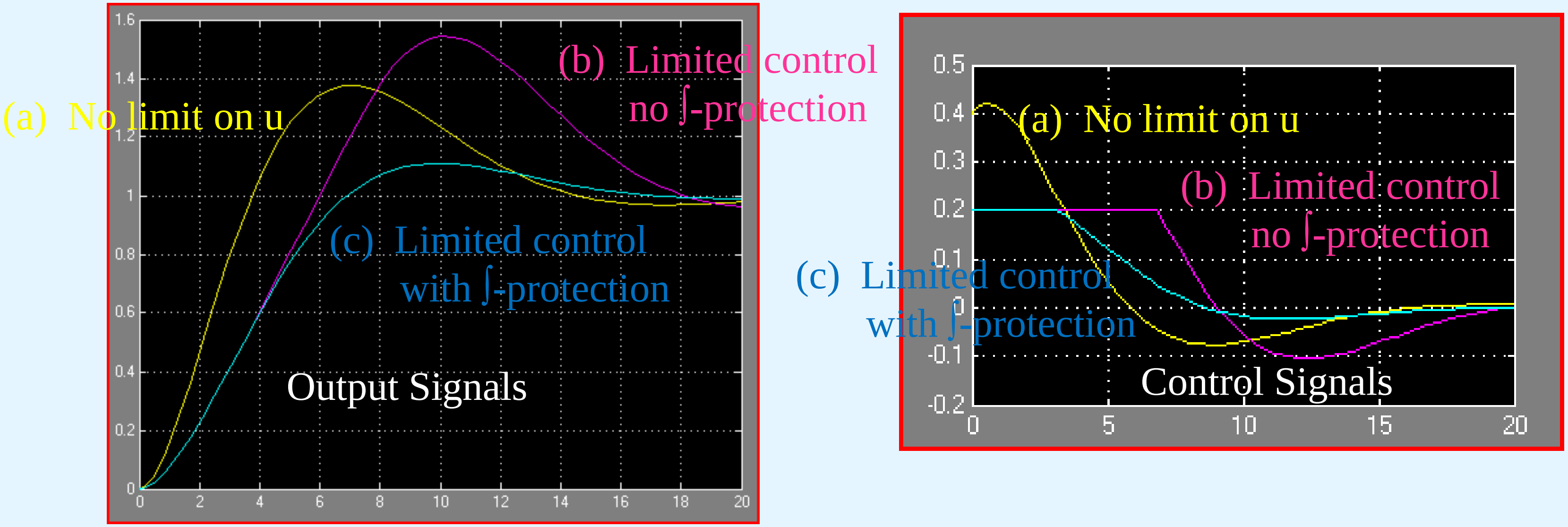
Example (Aström and Wittenmark)

- Lack of integral protection will often result in large overshoots in system response.
  - Since long periods of + (or –) e will cause UI to build up large values. Then e reverses…
- <u>Ex</u>. A motor with transfer function G(s) = 1/s(s+1) is to be controlled using a PI controller*

$$u(s) = K\left[1 + \frac{1}{T_1 s}\right] e(s)$$

with $K = 0.4$, $T_1 = 5$ sec

  - Examine step response when $|u(t)| \le 0.2$, with and without integral windup protection.



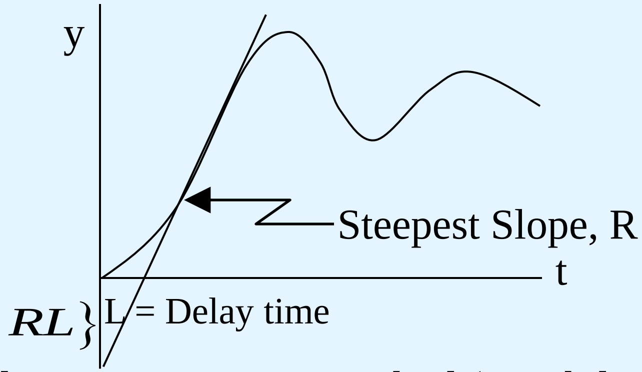
* Note: The I part of the controller is not really needed here since G(s) contains a 1/s. But it is only an example.

PID Initial Tuning Rules

- Ziegler-Nichols tuning formulas (1942). Can be used on a physical process directly.
- Although no need to model G(s), the formulas are based on $G(s) = \frac{k_g e^{-sL}}{1+sT}$ ...*FOPDT* model

Transient Response Method (Reaction Curve Method)

Obtain unit step response of open-loop system. [G(s) must be open-loop stable].



	K	T_1	T_2
P	1/RL	-	-
PI	0.9/RL	3L	-
PID	1.2/RL	2L	0.5L

$R = k_g / T$ for first order plus dead time (FOPDT) model

Ultimate Sensitivity Method (Instability Method of Ziegler-Nichols)

1. Use a P controller (u=Ke) to stabilize system.
2. Slowly increase gain K until the system is on the stability boundary $\Rightarrow K_{max}$.
3. Obtain time period of oscillations, $T_p = 2\pi/\omega_p \Rightarrow \angle K_{max} G(j\omega_p) = -180^0$ and $|K_{max} G(j\omega_p)| = 1$.

	K	T_1	T_2
P	$0.5K_{max}$	-	-
PI	$0.45K_{max}$	$T_p/1.2$	-
PID	$0.6K_{max}$	$T_p/2$	$T_p/8$

- A "guideline" for selecting sampling interval, h

$h \sim 0.03T_p$ to $0.05T_p$ (20-30 times max frequency)

Recent Tuning Methods

- Find the best-fit FOPDT model to a plant transfer function (must be open-loop stable)

$$\min_{L,T} J = \int_0^\infty |t^\alpha [g(t) - \frac{G(0)}{T} e^{-(t-L)/T} U(t-L)]|^p \, dt; R = K/T = G(0)/T; g(t) = \textit{impulse response of OL system}$$

  - $\alpha = 0$ and $p = 2 \Rightarrow$ Integral squared error (ISE)
  - $\alpha = 0$ and $p = 1 \Rightarrow$ Integral absolute error (IAE)
  - Use Ziegler-Nichols tuning formulas using identified parameters.

- Frequency Response Method

  Get $K_{\max}$ (=gain margin) and ω_p from the Bode plot of $G(j\omega)$ where $\angle G(j\omega_p) = -180^0$.

  Evidently, $k_g = G(0) = dc\ gain$

  From the FOPDT model to be matched, $G(0)e^{-j\omega_p L}/(1 + j\omega_p T) = -1/K_{\max} + j0$

$$\frac{G(0)[\cos\omega_p L - \omega_p T \sin\omega_p L]}{1 + (\omega_p T)^2} = -\frac{1}{K_{\max}}$$

$$\sin\omega_p L + \omega_p T \cos\omega_p L = 0$$

Use Ziegler-Nichols tuning formulas using identified parameters.

- Match first and second order derivatives of *G(s)*

Can show that if $G_a(s) = \frac{k_g e^{-sL}}{1+Ts} \Rightarrow \frac{dG_a/ds}{G_a(s)} = -L - \frac{T}{1+Ts}; \frac{d^2G_a/ds^2}{G_a(s)} - \left(\frac{dG_a/ds}{G_a(s)}\right)^2 = \left(\frac{T}{1+Ts}\right)^2$

$$So, L + T = -\frac{dG_a/ds}{G_a(s)}\Big|_{s=0} \approx -\frac{dG/ds}{G(s)}\Big|_{s=0}; T^2 + (L+T)^2 = \frac{d^2G_a/ds^2}{G_a(s)}\Big|_{s=0} \approx \frac{d^2G/ds^2}{G(s)}\Big|_{s=0}; k_g = G(0)$$

Shaping Closed-loop Transfer Function

- Guillemin-Truxal Procedure

$$T(s)=\frac{G(s)H(s)}{1+G(s)H(s)} \Rightarrow H(s)=\frac{T(s)}{1-T(s)}\cdot\frac{1}{G(s)}$$

- (#poles -#zeros) of $T(s) \geq$ (#poles -#zeros) of $G(s)$
- $G(s)$ must be stable and minimum phase

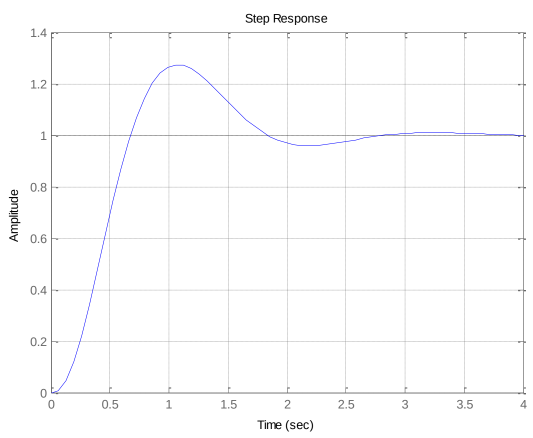
- Example

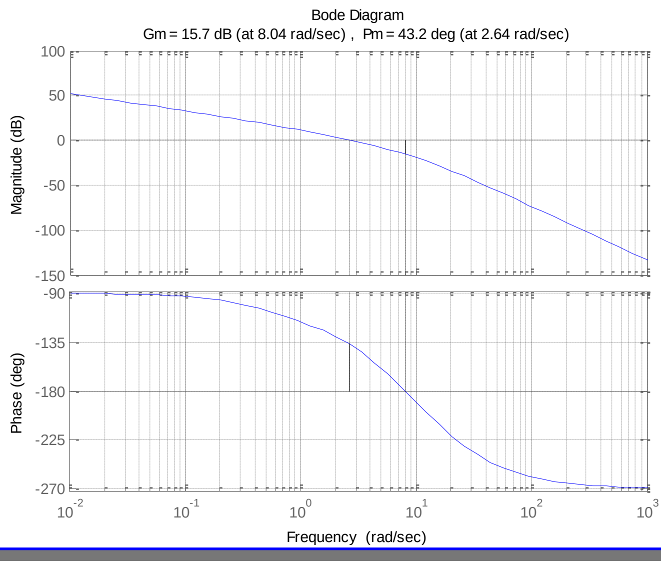
$$G(s)=\frac{4}{s(s+1)(s+5)}$$

$$Want\ T(s)=\frac{210(s+1.5)}{(s+1.75)(s+16)(s^2+3s+11.25)}$$

$$H(s)=\frac{T(s)}{1-T(s)}\frac{1}{G(s)}=\frac{52.5(s+1.5)}{(s+14.86)}$$ after cancelling terms

Lead Compensation





Inverse-based Controller & Disturbance Rejection

- Fix Loop Gain
  - Recall like to have -20 dB slope near cross-over. So, select $LG_{ain}=\omega_c/s \Rightarrow H(s)\approx G^{-1}(s)\ \omega_c/s$
  - Recall

$$y/d = G_d(s)/(1+LG_{ain}(s)) \Rightarrow |1+LG_{ain}(s)|>|G_d(s)| \Rightarrow |\underline{LG}_{ain}(s)| \approx |G_d(s)| \text{ near } \omega_c$$

$$\Rightarrow |H(s)|>|H_{\min}(s)| \approx |G^{-1}(s)G_d(s)|$$

  - For disturbance rejection in the steady state, need a zero at s=0

$$so,\ H(s)=K(1+\frac{1}{T_i s})G^{-1}(s)G_d(s)$$

- Example (Skogestad & Postelthwaite)

$$G(s)=\frac{200}{(10s+1)}\frac{1}{(0.05s+1)^2};\ G_d(s)=\frac{100}{(10s+1)}$$

(i) From $|\underline{LG}_{ain}(s)| \approx G_d(s)=\dfrac{100}{(10s+1)}$

$$\Rightarrow H(s)\approx G^{-1}(s)G_d(s)=\frac{1}{2}(0.05s+1)^2 \approx \frac{1}{2}$$

(ii) $H(s)=\dfrac{1}{2}(1+\dfrac{1}{T_i s});\ \dfrac{1}{T_i}\approx 0.1\omega_c \Rightarrow T_i \approx 1/0.6$

$\phi_m \approx 24^0$... too small

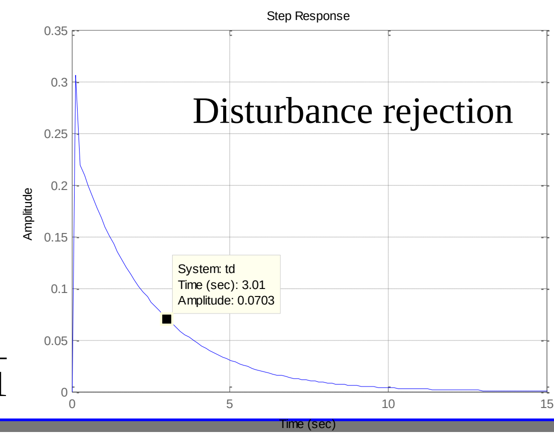
(iii) add lead to $H(s)$ and increase gain $\Rightarrow H(s)=(1+\dfrac{0.6}{s})\dfrac{0.05s+1}{0.005s+1}$

Specs:

rise time, $t_r = \dfrac{1.8}{\omega_c} \le 0.3\text{sec} \Rightarrow \omega_c \ge 6\text{rad/sec}$

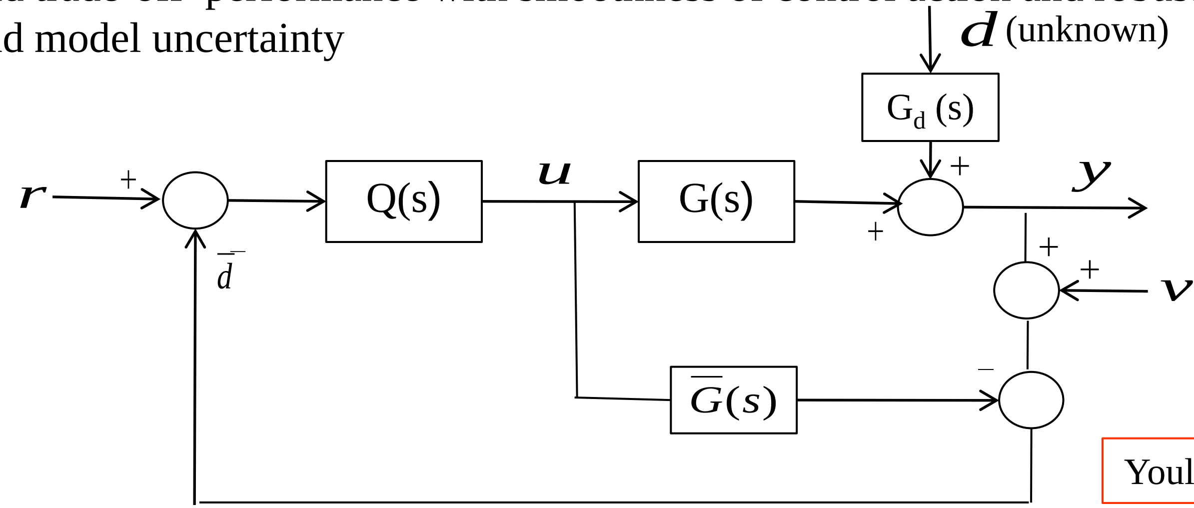
overshoot $\le 5\% \Rightarrow \zeta \approx 0.7$

$|y_d(t)| \le 0.1$ after 3 seconds $\Rightarrow \text{Re}(p) \le -1$



Internal Model Control (IMC)

- Two-step process
  - **Nominal Performance**: Design $\tilde{Q}(s)$ to yield optimal tracking and disturbance rejection (ignore m/s noise and model uncertainty)
  - **Robust Stability and Performance:** Use an IMC filter $f(s)$ so that $Q(s)=\tilde{Q}(s)f(s)$ is proper and trade-off performance with smoothness of control action and robustness to m/s noise and model uncertainty



Youla Parameterization

$$\text{Recall } y(s)=T(s)[r(s)-v(s)]+G_d(s)S(s)d(s)$$

$$T(s)=\frac{G(s)Q(s)}{1+Q(s)[G(s)-\bar{G}(s)]}\Rightarrow Q(s)=\frac{H(s)}{1+\bar{G}(s)H(s)}\ \ or\ \ H(s)=\frac{Q(s)}{1-\bar{G}(s)Q(s)}$$

$$When\ \bar{G}(s)=G(s),\ T(s)=G(s)Q(s)\ and\ S(s)=1-G(s)Q(s)\Rightarrow Q(s)=H(s)S(s)=H(s)[1-T(s)]$$

$$\Rightarrow H(s)=\frac{Q(s)}{1-T(s)}=\frac{1}{G(s)}\left(\frac{T(s)}{1-T(s)}\right)\ldots\ \text{Guillemin}-\text{Truxal Procedure}$$

IMC Design Process

- **Design for Nominal Performance:**
  1. Factor the OL system model into an invertible minimum-phase part G_m (s) and a a non-invertible all-pass part G_a (s)

$$G(s) = G_m(s)G_{nm}(s) = G_m(s)G_a(s); \quad G_{nm}(s) = e^{-sL}\prod_i \frac{-s+z_i}{s+z_i} = G_a(s)$$

  2. Let T(s) = f (s) G_a (s) ⇒ Q(s) = $T(s)/G(s)$ = f (s)/ G_m (s) ⇒ H(s) = Q(s)/[1-T(s)]

$$H(s) = \frac{f(s)}{G_m(s)}\frac{1}{1-f(s)G_a(s)} = \frac{1}{G_m(s)}\frac{1}{f^{-1}(s)-G_a(s)}$$

$$f(s) = \frac{1+\beta s}{(1+\lambda s)^n} \text{ for tracking steps} \Rightarrow f(0)=1$$

$$f(s) = \frac{1+n\lambda s}{(1+\lambda s)^n} \text{ for tracking ramps} \Rightarrow f(0)=1, df/ds|_{s=0}=0$$

n is selected to make $Q(s)$ proper.

- **Design for Robust Stability and Robust Performance:**

Recall from RS discussion, $\left|\frac{G(j\omega)-\bar{G}(j\omega)}{\bar{G}(j\omega)}\right| \le w_T(\omega) \Rightarrow |T(j\omega)| \le \frac{1}{w_T(\omega)}$

Pick λ (and β for tracking steps) to satisfy RS constraints.

IMC Design Examples - 1

Example 1: consider a minimum phase system given by

$$G(s) = \frac{1000}{s(s+10)} = G_m(s) \Rightarrow G_a(s) = 1$$

$$Want: (i) K_v \geq 100; (ii) 20\log_{10}|G(j\omega)H(j\omega)| > 34dB \;\; for\; \omega < 1\, rad/\sec, (iii)\; \phi_m \geq 45^0$$

$$(iv)\; 20\log_{10}|G(j\omega)H(j\omega)| < -26dB \;\; for\; \omega > 100\; rad/\sec$$

To track steps, select $T(s) = f(s) = \frac{1+\beta s}{(1+\lambda s)^3}; \beta = 3\lambda$ to track ramps

$$Q(s) = \frac{s(s+10)(1+\beta s)}{1000(1+\lambda s)^3}$$

$$\Rightarrow H(s) = \frac{s(s+10)(1+\beta s)}{1000[(1+\lambda s)^3 - (1+\beta s)]} = \frac{(s+10)(1+\beta s)}{1000\lambda^3[s^2 + \frac{3}{\lambda}s + \frac{1}{\lambda^3}(3\lambda - \beta)]}$$

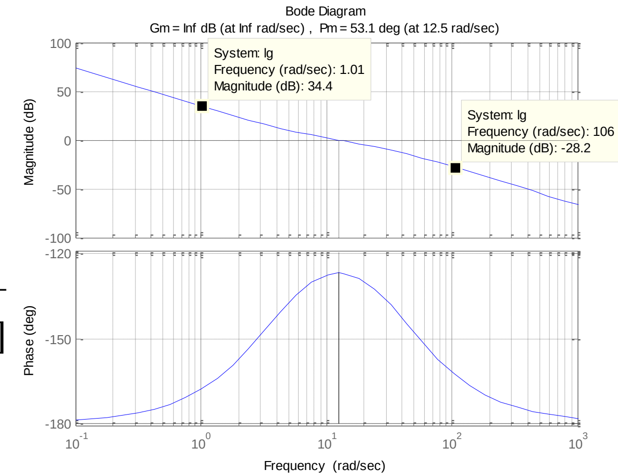
$$Design\,1: \lambda = 0.1\; and\;\; \beta = 0.3 \Rightarrow H(s) = \frac{(s+10)(1+0.3s)}{s(s+30)}$$

$$\omega_c = 10\, rad/\sec; \phi_m = 53.1^0; 20\log_{10}|G(j\omega)H(j\omega)|_{\omega=1} = 30.1dB; 20\log_{10}|G(j\omega)H(j\omega)|_{\omega=100} = -31.1dB$$

$$Design\,2: \lambda = 0.08\; and\;\; \beta = 0.24 \Rightarrow H(s) = \frac{1.9531(s+10)(1+0.24s)}{s(s+37.5)}$$

$$\omega_c = 12.5\, rad/\sec; \phi_m = 53.1^0; 20\log_{10}|G(j\omega)H(j\omega)|_{\omega=1} = 34.4dB; 20\log_{10}|G(j\omega)H(j\omega)|_{\omega=100} = -28.2dB$$

Design 2



IMC Design Examples - 2

Example 2: (ideal PID controller) consider a second order system with transport delay

$$G(s)=\frac{\omega_n^2 e^{-s\tau}}{s^2+2\zeta\omega_n+\omega_n^2} \Rightarrow G_m(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2} \; and \; G_a(s)=e^{-s\tau}$$

To track steps, select $T(s)=f(s)G_a(s)=\dfrac{e^{-s\tau}}{(1+\lambda s)}$

$$H(s)=\frac{1}{G_m(s)}\frac{1}{[f(s)]^{-1}-G_a(s)}=\frac{s^2+2\zeta\omega_n+\omega_n^2}{\omega_n^2}\cdot\frac{1}{1+\lambda s-e^{-s\tau}}\approx\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}\cdot\frac{1}{(\lambda+\tau)s}$$

$$\Rightarrow H(s)=\frac{2\zeta}{\omega_n(\lambda+\tau)}+\frac{1}{(\lambda+\tau)s}+\frac{s}{\omega_n^2(\lambda+\tau)}$$

Ideal PID controller

$$\approx\frac{2\zeta}{\omega_n(\lambda+\tau)}[1+\frac{1}{(2\zeta/\omega_n)s}+\frac{1}{2\zeta\omega_n}s/\left(\frac{1}{1+s/2\zeta\omega_n N}\right)]$$

makes the controller causal

IMC Design Examples - 3

Example 3: (Distillation column reboiler) consider *a non-minimum phase* system given by

Chapter 10, section 7 by Braatz in Levine, 1996

$$G(s)=\frac{-3s+1}{s(s+1)}$$

$$Want\ \left|\frac{G(j\omega)-\bar{G}(j\omega)}{\bar{G}(j\omega)}\right|\le\left|\frac{2j\omega+0.2}{j\omega+1}\right|\Rightarrow|T(j\omega)|\le\left|\frac{j\omega+1}{2j\omega+0.2}\right|$$

$$Know\ G_m(s)=\frac{3s+1}{s(s+1)}\ and\ G_a(s)=\frac{-3s+1}{3s+1}$$

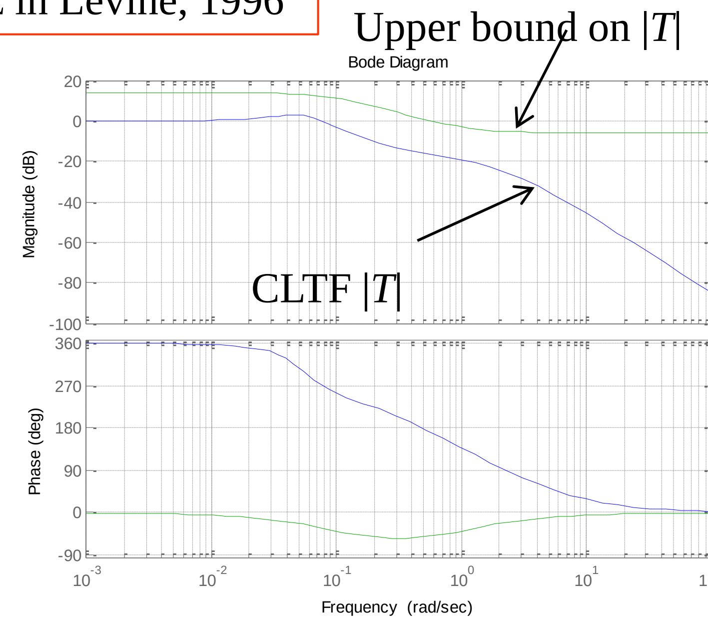
To track ramps, select $T(s)=f(s)G_a(s)=\dfrac{(-3s+1)(3\lambda s+1)}{(3s+1)(1+\lambda s)^3}$

$$Q(s)=\frac{T(s)}{G(s)}=\frac{f(s)}{G_m(s)}=\frac{s(s+1)(3\lambda s+1)}{(3s+1)(1+\lambda s)^3}$$

$$H(s)=\frac{1}{G_m(s)}\frac{1}{[f(s)]^{-1}-G_a(s)}=\frac{s(s+1)}{3s+1}\cdot\frac{1}{\dfrac{(1+\lambda s)^3}{(3\lambda s+1)}-\dfrac{-3s+1}{3s+1}}$$

$$\Rightarrow H(s)=\frac{(3\lambda s+1)(s+1)}{3\lambda^3s^3+(9\lambda+\lambda^2)\lambda s^2+(9+12\lambda)\lambda s+6}$$

Plot of $T(s)$ for λ=5.4



IMC Design Examples - 4

Example 4: Generic IMC procedure on unstable systems leads to unacceptable overshoots in step response and resonant peaks.

Consider an unstable system given by

$$G(s)=\frac{6}{s-2}\Rightarrow G(s)=G_m(s)G_a(s)\ where\ G_m(s)=\frac{6}{s-2};G_a(s)=1$$

Let $T(s)=f(s)=\frac{(1+\alpha s)}{(1+\lambda s)^2}$.

Need $T(2)=1$ (Recall $GH=\infty$ @ a pole of $G(s)$)

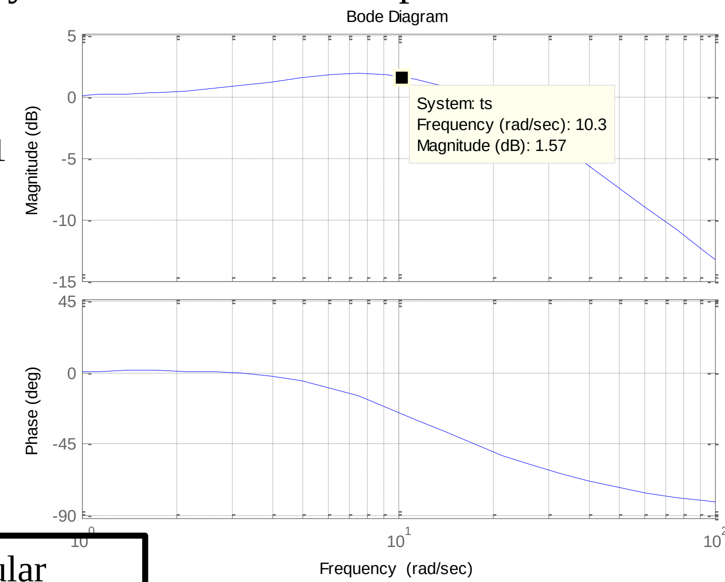
$$\Rightarrow \alpha=2\lambda+2\lambda^2=2\lambda(1+\lambda)$$

$$Q(s)=\frac{T(s)}{G(s)}=\frac{(2\lambda(1+\lambda)s+1)(s-2)}{6(1+\lambda s)^2}$$

$$H(s)=\frac{Q(s)}{1-G(s)Q(s)}=\frac{1}{G_m(s)}\frac{1}{[f(s)]^{-1}-G_a(s)}$$

$$=\frac{(s-2)}{6}\cdot\frac{1}{\left[\frac{(1+\lambda s)^2}{(2\lambda(1+\lambda)s+1)}-1\right]}$$

$$\Rightarrow H(s)=\frac{[2\lambda(1+\lambda)s+1]}{6\lambda^2 s}\Rightarrow LG=\frac{[2\lambda(1+\lambda)s+1]}{\lambda^2 s(s-2)}$$



Do a regular Bode design instead (Gain compensation or Lead Design)

$$f(s)=\frac{\prod_{i=1}^{q}(1+\alpha_i s)}{(1+\gamma s)^q(1+\lambda s)(1+0.1\lambda s)^m}$$

$q=$ No. of unstable poles

m used to make $Q(s)$ proper

Problems:

- Generic IMC procedure gives resonant peak at $1/\lambda \Rightarrow$ overshoot in step response
- Select $1/\lambda = 5p$ (pole location), closed-loop BW is approximately 28 rad/sec $\cong 3/\lambda$
- How to get rid of resonance: Use a different filter (see Campi, Lee and Anderson, *Int. J. of Nonlinear and Robust Control*, Vol. 4, pp. 757-775, 1994.). There exist better methods.

Weighted Sensitivity & IMC - 1

- Recall Youla Parameterization $H(s) = Q(s)[I - G(s)Q(s)]^{-1} = Q(s)S^{-1}(s) \Rightarrow Q(s) = H(s)S(s)$
- For stable and proper transfer functions, one can define a transfer function

$$T(s) = f(s) = \frac{1}{(\lambda s+1)^k} \; or \; \frac{1+k\lambda s}{(\lambda s+1)^k}; k \geq 1 = G(s)Q(s) \Rightarrow Q(s) = G^{-1}(s)T(s)$$

- So, weighted sensitivity $W_s(s)S(s) = W_s(s)[I - G(s)Q(s)] = W_s(s)[I - T(s)]$
- One can show (Doyle et al., Chapter 10) that as $\lambda \rightarrow 0$

$$\lim_{\lambda \to 0} \| G(j\omega)[I - T(j\omega)] \|_\infty = \max_{\omega} \lim_{\lambda \to 0} | G(j\omega)[I - T(j\omega)] | = 0$$

$$Can \; find \; \lambda \ni \| W_s(j\omega)S(j\omega)] \|_\infty = \| W_s(j\omega)[I - T(j\omega)] \|_\infty < 1$$

Recall Nominal performance constraint

Idea of proof: $for \; small \; \omega \leq \omega_1, |T(j\omega)| \approx 1 \Rightarrow |1 - T(j\omega)| \approx \varepsilon \Rightarrow \max_{\omega \leq \omega_1} |G(j\omega)(1 - T(j\omega))| \leq \varepsilon \| G \|_\infty$

$for \; large \; \omega > \omega_1, \max_{\omega > \omega_1} |G(j\omega)(1 - T(j\omega))| \leq 2 \max_{\omega > \omega_1} |G(j\omega)|$

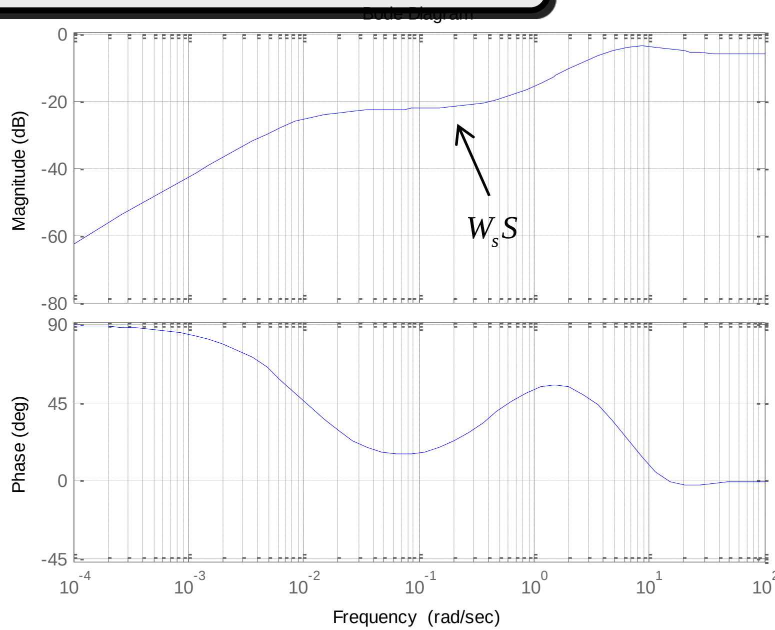
so, by selecting λ sufficiently small, we can make ε small and $\max_{\omega > \omega_1} |G(j\omega)|$ small.

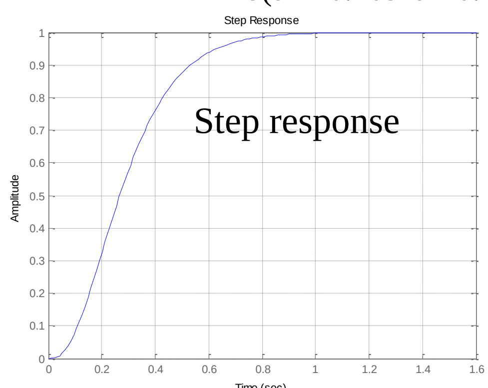
Design Procedure:

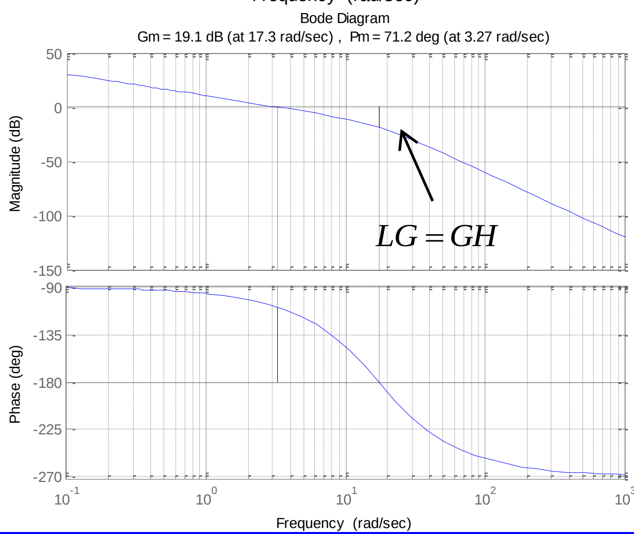
- Given a weighting matrix $W_s(s)$ and $G(s)$
- Set k = relative degree of G (s) = degree of denominator of $G(s)$
- Choose λ so that $\| W_s(s)S(s) \|_\infty < 1$
- Set $Q(s) = G^{-1}(s)T(s)$
- Set $H(s) = Q(s)[I - G(s)Q(s)]^{-1}$

Weighted Sensitivity & IMC Example - 2

- **Example:**
  - $G(s)=\dfrac{1}{s+1}+\dfrac{0.1s}{s^2+.02s+0.25}=\dfrac{1.1s^2+0.12s+0.25}{s^3+1.02s^2+0.27s+0.25}$
  - $W_s(s)=0.5\dfrac{(s+0.5)}{(s+0.01)}$
  - $k=3\Rightarrow T(s)=\dfrac{1}{(\lambda s+1)^3}$
  - $\lambda=0.1\Rightarrow S(s)=\dfrac{0.001s^3+0.03s^2+0.3s}{0.001s^3+0.03s^2+0.3s+1}$
  - $W_s(s)S(s)=\dfrac{0.0005s^4+0.01525s^3+0.1575s^2+0.075s}{0.001s^4+0.03001s^3+0.3003s^2+1.003s+0.01}$
  - $\|W_sS\|_\infty=0.6428$
  - $Q(s)=G^{-1}(s)T(s)=\dfrac{909.09(s+1)(s^2+0.02s+0.25)}{(s+10)^3(s^2+0.1091s+0.2273)}$
  - $H(s)=Q(s)[1-G(s)Q(s)]^{-1}=\dfrac{909.09(s+1)(s^2+0.02s+0.25)}{s(s^2+0.1091s+0.2273)(s^2+30s+300)}$







Co-prime Factorization: Unstable & Non-minimum Phase Systems - 1

- For unstable and/or non-minimum phase systems, inversion leads to non-minimum phase and/or unstable $Q(s)$. Need a generalization in this case.

If $G(s)$ is given (not necessarily stable or minimum phase), then it can be written as $G(s) = N(s)M^{-1}(s)$ where $N(s)$ and $M(s)$ are (co-prime) transfer functions $\Rightarrow$ No pole-zero cancellation, proper & stable

If $H(s) = \dfrac{N_H(s)}{M_H(s)} \Rightarrow$ *characteristic polynomial* $: 1 + G(s)H(s)$

For stability, need poles in LHP $\Rightarrow 1 + G(s)H(s) \neq 0 \forall s \in RHP$

$\Rightarrow M_H(s)M(s) + N(s)N_H(s) \neq 0 \forall s \in RHP$

Suppose we find $X(s)$ and $Y(s)$ such that $X(s)N(s) + Y(s)M(s) = 1$ (called *Bezout* identity), where $N(s), M(s), X(s)$ and $M(s)$ are proper and stable. Then,

Then, $H(s) = \dfrac{N_H(s)}{M_H(s)} = \dfrac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)}$ is such that the closed-loop system is stable.

Also, $M_H(s)M(s) + N(s)N_H(s) = Y(s)M(s) - N(s)Q(s)M(s) + N(s)X(s) + N(s)M(s)Q(s) = 1$

$\Rightarrow H(s)$ is co-prime as well $\Rightarrow$ stable and proper

Note $S(s) = (1 + \dfrac{N}{M} . \dfrac{X + MQ}{Y - NQ})^{-1} = M(Y - NQ)$. $\Rightarrow \min_Q \| W_s M(Y - NQ) \|_\infty$

For unstable systems, can approximately minimze $\min_Q \| W_s MY(1 - T(s)) \|_\infty$

Co-prime Factorization: Unstable & Non-minimum Phase Systems - 2

- Co-prime factorization of $G(s)$ is easy, but solving Bezout identity is not.

If $G(s)$ is given (not necessarily stable or minimum phase), then it can be written as $G(s) = N(s)M^{-1}(s)$ where $N(s)$ and $M(s)$ are (co-prime) transfer functions $\Rightarrow$ No pole-zero cancellation, proper & stable

Examples: Getting co-prime factorization of $G(s)$ is easy.

- If $G(s)$ is stable and proper, $N(s) = G(s), M(s) = 1$
- If $G(s)$ is unstable and proper, divide numerator and denominator by a common stable polynomial.

Example: $G(s) = \dfrac{1}{(s-1)(s-2)} \Rightarrow N(s) = \dfrac{1}{(s+\lambda)^2}, M(s) = \dfrac{(s-1)(s-2)}{(s+\lambda)^2}$

- If $G(s)$ is unstable and non-minimum phase, leave non-minimum phase part in $N(s)$ and divide numerator and denominator by a common stable polynomial.

Example: $G(s) = \dfrac{(s-1)}{s(s-2)} \Rightarrow N(s) = \dfrac{(s-1)}{(s+\lambda)^2}, M(s) = \dfrac{s(s-2)}{(s+\lambda)^2}$

Co-prime Factorization via State Space Methods - 1

- State feedback and observer feedback allows us to compute co-prime factorization and the solution of Bezout identity rather easily
- Given $G(s)$, find state space representation (e.g., SCF, SOF, minimal, balanced). Valid for MIMO systems as well.

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$
$$\underline{y} = C\underline{x} + D\underline{u}$$

$$G(s) = C(sI - A)^{-1}B + D \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

- Choose a feedback matrix, K such that A-BK is stable. So, if we define signals

$$\underline{v} = \underline{u} + K\underline{x} \Rightarrow \dot{\underline{x}} = (A - BK)\underline{x} + B\underline{v} \text{ and } \underline{u} = -K\underline{x} + \underline{v}$$
$$\dot{\underline{x}} = (A - BK)\underline{x} + B\underline{v};\ \underline{y} = (C - DK)\underline{x} + D\underline{v}$$

$$so,\ \underline{u}(s) = M(s)\underline{v}(s) \Rightarrow M(s) = I_m - K(sI - A + BK)^{-1}B \triangleq \begin{bmatrix} A - BK & B \\ -K & I_m \end{bmatrix}$$

$$\underline{y}(s) = N(s)\underline{v}(s) \Rightarrow N(s) = D + (C - DK)(sI - A + BK)^{-1}B \triangleq \begin{bmatrix} A - BK & B \\ C - DK & D \end{bmatrix}$$

$$\Rightarrow G(s) = N(s)M^{-1}(s)$$

Co-prime Factorization via State Space Methods - 2

- Choose a feedback matrix L such that A-LC is stable.

Recall observer equation: $\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + B\underline{u} + L(\underline{y} - C\hat{\underline{x}} - D\underline{u})$

$\Rightarrow \dot{\hat{\underline{x}}} = (A-LC)\hat{\underline{x}} + L\underline{y} + (B-LD)\underline{u}$

Split observer equations into two parts (recall superposition): $\hat{\underline{x}} = \hat{\underline{x}}_1 + \hat{\underline{x}}_2$

$\dot{\hat{\underline{x}}}_1 = (A-LC)\hat{\underline{x}}_1 + L\underline{y}; \underline{v}_1 = K\hat{\underline{x}}_1 \Rightarrow \underline{v}_1(s) = X(s)\underline{y}(s)$

$\Rightarrow X(s) = K(sI - A + LC)^{-1}L \triangleq \begin{bmatrix} A-LC & L \\ K & 0 \end{bmatrix}$

$\dot{\hat{\underline{x}}}_2 = (A-LC)\hat{\underline{x}}_2 + (B-LD)\underline{u}; \underline{v}_2 = \underline{u} + K\hat{\underline{x}}_2 \Rightarrow \underline{v}_2(s) = Y(s)\underline{u}(s)$

$\Rightarrow Y(s) = I_m + K(sI - A + LC)^{-1}(B-LD) \triangleq \begin{bmatrix} A-LC & B-LD \\ K & I_m \end{bmatrix}$

$$\begin{aligned} \underline{u}(s) &= M(s)\underline{v}(s) \\ \underline{y}(s) &= N(s)\underline{v}(s) \\ \underline{v}_1(s) &= X(s)\underline{y}(s) \\ \underline{v}_2(s) &= Y(s)\underline{u}(s) \\ \underline{v}_1(s) + \underline{v}_2(s) &= \underline{v}(s) \end{aligned}$$

Evidently, $\underline{v}_1 + \underline{v}_2 = \underline{u} + K\hat{\underline{x}} = \underline{v}$ in the steady state

Look at signals now: $\underline{v}_1(s) = X(s)\underline{y}(s) = X(s)N(s)\underline{v}(s)$

$$\underline{v}_2(s) = Y(s)\underline{u}(s) = Y(s)M(s)\underline{v}(s)$$

$$\text{so, } \underline{v}_1(s) + \underline{v}_2(s) = [X(s)N(s) + Y(s)M(s)]\underline{v}(s) = \underline{v}(s)$$

$$\Rightarrow X(s)N(s) + Y(s)M(s) = I_m$$

Design Examples - 1

- **Example 2**: Minimize weighted sensitivity $\|W_s S\|_\infty$ for $G(s)=\frac{1}{(s-2)^2}; W_s(s)=\frac{100}{(s+1)}$
- State space equation (SCF)

$$\dot{\underline{x}}=\begin{bmatrix}0 & 1\\ -4 & 4\end{bmatrix}\underline{x}+\begin{bmatrix}0\\ 1\end{bmatrix}u;\ y=\begin{bmatrix}1 & 0\end{bmatrix}\underline{x}$$

- Find gains *K* such that *A-BK* has poles in LHP. Let us place poles at -1+j and -1-j. $K=[-2 \quad 6]$
- Find *N*(s) and *M* (s)

$$N(s)=C(sI-A+BK)^{-1}B=\frac{1}{s^2+2s+2};\ M(s)=1-K(sI-A+BK)^{-1}B=\frac{s^2-4s+4}{s^2+2s+2}$$

- Find observer gains *L* such that *A-LC* has poles in LHP. Place poles at -2+2j and -2-2j. Twice as fast as controller. $L=[8 \quad 36]$
- Find *X*(*s*) and *Y*(*s*)

$$X(s)=K(sI-A+LC)^{-1}L=\frac{200(s-1)}{s^2+4s+8};\ Y(s)=1+K(sI-A+LC)^{-1}B=\frac{s^2+10s+54}{s^2+4s+8}$$

- Select $T(s)=\frac{1}{(\lambda s+1)^2}\Rightarrow S(s)=\frac{\lambda^2 s^2+2\lambda s}{(\lambda s+1)^2}$

$Note: X(s)N(s)+Y(s)M(s)=I_m$

- Choose λ so that the infinity norm of $\|W_s YM(1-T)\|_\infty<1$

λ	Norm
0.1	222.49
0.01	22.63
0.001	2.26
0.0003	0.679

:

$$Q(s)=Y(s)N^{-1}(s)T(s)=\frac{1.1\text{x}10^7(s^2+2s+2)(s^2+10s+54)}{(s+3333)^2(s^2+4s+8)}$$

$$H(s)=\frac{X(s)+M(s)Q(s)}{Y(s)-N(s)Q(s)}=\frac{1.1\text{x}10^7(s^2+1.921s+1.911)(s^2+4.079s+8.373)}{s(s+6667)(s^2+10s+54)}$$

$\phi_m=76.3^0\ @1.62 rad/\sec$

Design Examples - 2

- **Example 2**: Minimize weighted sensitivity $\| W_s S \|_\infty$ for

$$G(s) = \frac{-6.475s^2 + 4.0302s + 175.77}{5s^4 + 3.5682s^3 + 139.5021s^2 + 0.0929s + 10^{-6}}$$

poles: 0, -0.0007, -0.3565 ± 5.27 *j*; *zeros* : −4.9081, 5.5308

settling time $\approx$ 8sec; % overshoot $\leq 10\% \Rightarrow \zeta = 0.6; \omega_n = 0.96 \Rightarrow T(s) \approx \frac{1}{s^2 + 1.2s + 1}$

$$\Rightarrow S(s) = \frac{s(s+1.2)}{s^2 + 1.2s + 1} \Rightarrow W_s(s) \approx \frac{s^2 + 1.2s + 1}{(s+0.001)(s+1.2)(0.001s+1)}$$ stable and strictly proper

- Since $G(s)$ is stable, $N(s) = G(s), M(s) = 1, X(s) = 0, Y(s) = 1$ so that $NX + MY = 1$

- Find Q_{im} (not necessarily proper) such that $\| W_s M(Y - NQ_{im}) \|_\infty = \| W_s(1 - GQ_{im}) \|_\infty$ is minimum

  Recall at RHP zero, $G=0 \Rightarrow | W_s(5.5308) | = 1.0210 \Rightarrow$ set $W_s = \frac{0.9}{1.021} W_s = \frac{0.8815s^2 + 1.058s + 0.8815}{(s+0.001)(s+1.2)(0.001s+1)}$

- From $| W_s(5.5308) | = 0.9, Q_{im} = \frac{-0.001021s^5 - 0.01814s^4 + 0.0586s^3 + 1.015s^2 - 4.1s + 3.995}{s^2 + 1.22s + 1} = \frac{W_s(s) - 0.9}{W_s(s)G(s)}$

- Select $J(s) = \frac{1}{(\lambda s + 1)^2}$ *and* $\min_{\lambda} \| W_s(1 - GQ_{im}J) \|_\infty$

$$Q(s) = Q_{im}(s)J(s) = \frac{-0.001021s^5 - 0.01814s^4 + 0.0586s^3 + 1.015s^2 - 4.1s + 3.995}{6.4\text{x}10^{-5}s^5 + 0.004877s^4 + 0.1258s^3 + 1.149s^2 + 1.32s + 1}$$

$$H(s) = \frac{-15.95s^5 - 283.5s^4 + 915.6s^3 + 15860s^2 - 64060s + 62420}{s^5 + 76.2s^4 + 1982s^3 + 18300s^2 + 21030s + 19.14}$$

$\phi_m = 57.8^0$ @ 0.669 *rad* / sec
$\gamma_m = 18.9dB$ @ 3.08 *rad* / sec

λ	Norm
0.1	1.1199
0.08	1.0759
0.04	0.9880
0.02	0.9367