The University of Connecticut **Spring 2016**
Dept. of ECE **KRP**

ECE 6108: Linear Programming and Network Flows
Midterm (Due April 8, 2016)

1. (10 points) Suppose Z is a random variable taking values in the set $\{0,1, 2, \ldots, K\}$ with corresponding probabilities $\{p_0, p_1, \ldots, p_K\}$. We are given the first two moments of Z

$$E(Z) = \sum_{k=0}^{K} kp_k = b_1$$
$$E(Z^2) = \sum_{k=0}^{K} k^2 p_k = b_2$$

We would like to obtain upper and lower bounds on the value of the fourth moment of Z. Formulate this as two standard linear programming (SLP) problems, one for the lower bound and the other for the upper bound.

2. (20 points) Consider the linear programming problem

$$\begin{aligned} \min\ & x_1 + 3x_2 + 2x_3 + 2x_4 \\ s.t.\ & 2x_1 + 3x_2 + x_3 + x_4 = b_1 \\ & x_1 + 2x_2 + x_3 + 3x_4 = b_2 \\ & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_i \geq 0, i = 1,2,3,4 \end{aligned}$$

where b_1, b_2 are free parameters. Let $P(b_1, b_2)$ be the feasible set. Use the column geometry of linear programming to answer the following questions.

(a) Characterize explicitly (preferably with a picture in 2-D) the set of all (b_1, b_2) for which $P(b_1, b_2)$ is nonempty.
(b) Characterize explicitly (preferably with a picture) the set of all (b_1, b_2) for which some basic feasible solution is degenerate.
(c) There are four bases in this problem; in the i^{th} basis, all variables except for x_i are basic. For every (b_1, b_2) for which there exists a degenerate basic feasible solution, enumerate all bases that correspond to each degenerate basic feasible solution.
(d) For $i = 1,2,3,4$, let $S_i = \{(b_1, b_2) |$ the i^{th} basis is optimal$\}$. Identify, preferably with a picture, the sets S_i, $i = 1,2,3,4$.
(e) For which values of (b_1, b_2) is the optimal solution degenerate?
(f) Let $b_1 = 1.8$ and $b_2 = 1.4$. Suppose we start the simplex algorithm with (x_2, x_3, x_4) as the basic variables. Which path will the simplex method follow?

3. (20 points) Let A be a given matrix. Show that exactly one of the following two statements must be true.

   a. There exists some $\underline{x} \neq \underline{0}$ such that $A\underline{x} = \underline{0}$, $\underline{x} \geq \underline{0}$.
   b. There exists some $\underline{\lambda}$ such that $A^T\underline{\lambda} > \underline{0}$.

   (Hint: Consider the problem of maximizing $\underline{e}^T\underline{x}$ subject to constraints in part *a* and use duality to establish the contradiction between *a* and *b*).

4. (10 points) Consider the linear programming problem:

$$\begin{aligned} &\min\ x_1 + x_2 \\ &s.t.\ \ x_1 + 2x_2 = b \\ &\qquad x_i \geq 0, i = 1, 2 \end{aligned}$$

   (a) Find (by inspection) an optimal solution, as a function of b.
   (b) Draw a graph showing the optimal cost as a function of b.
   (c) Use the picture in part (b) to obtain the set of all dual optimal solutions, for every value of b.

5. (20 points) Consider a linear programming problem of the form:

$$\begin{aligned} &\min\ \underline{c}^T\underline{x} + \underline{d}^T\underline{y} \\ &s.t.\ \ A\underline{x} + D\underline{y} \leq \underline{b} \\ &\qquad F\underline{x} \leq \underline{f} \\ &\qquad \underline{y} \geq \underline{0} \end{aligned}$$

   (a) Suppose that we have access to a very fast subroutine for solving problems of the form

$$\begin{aligned} &\min\ \underline{h}^T\underline{x} \\ &s.t.\ \ F\underline{x} \leq \underline{f} \end{aligned}$$

   for arbitrary cost vectors $\underline{h}$. How would you go about decomposing the problem?

   (b) Suppose that we have access to a very fast subroutine for solving problems of the form

$$\begin{aligned} &\min\ \underline{d}^T\underline{y} \\ &s.t.\ \ D\underline{y} \leq \underline{h} \\ &\qquad \underline{y} \geq \underline{0} \end{aligned}$$

   for arbitrary right-hand side vectors $\underline{h}$. How would you go about decomposing the problem now?

(Hint: Use D-W for (a) and D-W on the dual for (b)).

6. (20 points) Consider a directed graph in which each arc is associated with a cost c_{ij}. For any directed cycle, we define its mean cost as the sum of the costs of its arcs, divided by the number of arcs. We are interested in a directed cycle whose mean cost is minimal. We assume that there exists at least one directed cycle in the directed graph.

   Consider the linear programming problem

$$\begin{aligned} &\max \mu \\ &s.t. \quad \lambda_i + \mu \le \lambda_j + c_{ij}, \text{ for all arcs } \langle i, j \rangle \end{aligned}$$

   (a) Show that the maximization problem is feasible.
   (b) Show that if (μ, $\underline{\lambda}$) is a feasible solution to the maximization problem, then the mean cost of every directed cycle is at least μ.
   (c) Show that the maximization problem has an optimal solution.
   (d) (Bonus 10 points) Show how an optimal solution to the maximization problem can be used to construct a directed cycle with minimal mean cost.